Physics 539 - Problem Set 2 - Due October 4, 2022

(1) In class, we discussed the metric

$$\mathrm{d}s^2 = -2e^q \mathrm{d}v\mathrm{d}u + g_{AB}(\mathrm{d}x^A + c^A\mathrm{d}v)(\mathrm{d}x^B + c^B\mathrm{d}v),$$

where q is independent of u at $v = 0$, but otherwise the functions q, g_{AB}, and c^A depend on all coordinates u, v, x^A. As discussed in class, this is a canonical form for the metric near a null hypersurface Y that is swept out by a family of orthogonal null geodesics from a codimension 2 spacelike submanifold W. W is the hypersurface $u = v = 0$ and Y is the hypersurface $v = 0$.

Calculate R_{uu} along Y, that is at $v = 0$. As explained in class, this step leads to Raychaudhuri's equation.

(2) Consider the metric

$$\mathrm{d}s^2 = (t^2 - 1)\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right),$$

where $\vec{x} = (x^1, \cdots, x^{D-1})$.

In this spacetime, consider the codimension 2 spacelike hypersurface W defined by $t = t_0$, $|\vec{x}| = R$, with constants t_0, R. What is the condition for W to be a trapped surface?

(3) In a spacetime M, let S be a spacelike hypersurface (dimension $D - 1$ if M has dimension D) and let $Q \subset S$ be a manifold with boundary, also of dimension $D - 1$; let ∂Q be the boundary of Q. For example, in Minkowski space, Q might be a closed ball and then its boundary ∂Q is a sphere. As usual, we let $J^+(Q)$ be the causal future of Q and $J^+(\partial Q)$ be the causal future of ∂Q. The boundaries of these sets are $\partial J^+(Q)$ and $\partial J^+(\partial Q)$.

Show that any point in $\partial J^+(Q)$ that is not in Q itself (in other words, any point that is strictly to the future of Q) is in $\partial J^+(\partial Q)$. This fact will be useful in discussing black holes.