Rigid cocycles & Real quadratic singular moduli (Part II)

PCMI Summer School 2022
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Last time: Discussed quadratic forms $\langle a, b, c\rangle := aX^2 + bXY + cY^2 \in \mathbf{Z}[X, Y]$

Collection of all $\mathrm{SL}_2(\mathbf{Z})$-orbits:

Let $\mathcal{F}_D$ be the set of primitive forms (with $a > 0$ if $D < 0$) of discriminant $D := b^2 - 4ac$. For any non-square discriminant D, there is a bijection

$$\begin{array}{ccc} \mathcal{F}_D / \mathrm{SL}_2(\mathbf{Z}) & \longrightarrow & \mathrm{Pic}^+\left(\mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]\right) \\ \langle a, b, c\rangle & \longmapsto & [(a, \frac{-b+\sqrt{D}}{2})] \end{array}$$

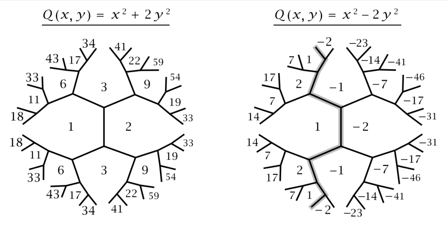
One single $\mathrm{SL}_2(\mathbf{Z})$-orbit:

For $\langle a, b, c\rangle$ defined reducedness:
When $D < 0$, by

$$|b| \le a \le c,$$
$$b \ge 0 \text{ if either equality holds.}$$

When $D > 0$, by

nearly reduced if $ac < 0$
reduced if $ac < 0$ and $b > |a + c|$.



Picture: Hatcher "Topology of numbers"

Additive cocycles

Consider $\mathbf{C}(z)$ = additive group of rational functions on $\mathbf{P}^1(\mathbf{C})$.
It is a left $\mathrm{GL}_2(\mathbf{Q})$-module for weight two action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) \ := \ (ad-bc)(-cz+a)^{-2} f\left(\frac{dz-b}{-cz+a}\right)$$

Consider *rational cocycles* := elements of $Z^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)\right)$
:= maps $\varphi : \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{C}(z)$ such that for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbf{Z})$

$$\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) + \gamma_1 \cdot \varphi(\gamma_2).$$

Toy example. Choose cusp $c = (r,s) \in \mathbf{P}^1(\mathbf{Q})$, and define

$$\begin{array}{cccc} p_c : & \mathrm{SL}_2(\mathbf{Z}) & \longrightarrow & \mathbf{C}(z), \\ & \gamma & \longmapsto & L(c) - L(\gamma c), \end{array} \qquad \text{where } \ L(c) := \frac{s}{sz-r}.$$

It is a cocycle, depends on c only up to coboundary. Why is it a cocycle?

1. Direct calculation
2. Recall that Eisenstein series $E_2(z)$ of weight two satisfies
$$E_2(z) \mid (\gamma - 1) \doteq p_\infty(\gamma^{-1})$$
3. An (invariant) *modular symbol* is a map $m : \mathbf{P}^1(\mathbf{Q}) \times \mathbf{P}^1(\mathbf{Q}) \longrightarrow M$ (where M = left $\mathrm{SL}_2(\mathbf{Z})$-module) such that
$$\begin{array}{rcl} m\{r,s\} & = & -m\{s,r\} \\ m\{r,t\} & = & m\{r,s\} + m\{s,t\} \end{array} \qquad \gamma \cdot m\{r,s\} = m\{\gamma r, \gamma s\}$$
For any cusp c, the map $\gamma \mapsto m\{c, \gamma c\}$ is a cocycle.

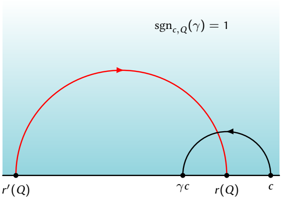
Knopp cocycle

More interesting examples were constructed by Knopp (1978).
Choose $c \in \mathbf{P}^1(\mathbf{Q})$ a cusp, and $F \in \mathcal{F}_D$ with $D > 0$ non-square. Define

$$\begin{array}{rccc} kn_{c,F} \; : & \mathrm{SL}_2(\mathbf{Z}) & \longrightarrow & \mathbf{C}(z) \\ & \gamma & \longmapsto & \displaystyle\sum_{Q \sim F} \frac{\mathrm{sgn}_{c,Q}(\gamma)}{z - r(Q)} \end{array}$$

where $r(\langle a, b, c\rangle) = \frac{-b+\sqrt{D}}{2a}$ is the first root and

$$\mathrm{sgn}_{c,Q}(\gamma) = \begin{cases} -1 & \text{if } Q(c) < 0 < Q(\gamma c) \\ 1 & \text{if } Q(c) > 0 > Q(\gamma c) \\ 0 & \text{otherwise} \end{cases}$$



One checks that it is well-defined and

- $kn_{c,F}$ is a cocycle.
- Its cohomology class $[kn_{c,F}] \in \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z))$ is independent of the choice of cusp c.

Natural choice of cusp is $c = \infty = (1, 0)$, get a parabolic cocycle kn_F determined by

$$kn_F(T) = 0, \qquad kn_F(S) = \sum_{Q \in \Sigma_F} \frac{\mathrm{sgn}\, Q(\infty)}{z - r(Q)}, \qquad \text{where } \Sigma_F := \{\langle a, b, c\rangle \sim F \; : \; ac < 0\}$$

Knopp cocycle

Example $F = \langle 1, 1, -1 \rangle$, discriminant $D = 5$

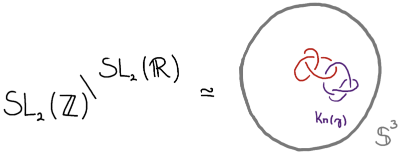
We have $\Sigma_F = \{\langle -1, 1, 1 \rangle, \langle -1, -1, 1 \rangle, \langle 1, -1, -1 \rangle, \langle 1, 1, -1 \rangle\}$ and

$$kn_F(S) = (z^2 - z - 1)^{-1} + (z^2 + z - 1)^{-1}.$$

Satisfies the identities

$$(1 + S) \cdot kn_F(S) = 0,$$
$$(1 + (ST) + (ST)^2) \cdot kn_F(S) = 0.$$

Application: Inspiration comes from linking numbers of modular geodesics.



If $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ is hyperbolic, get associated knot

$$\begin{aligned} \mathrm{Knot}(\gamma) &\hookrightarrow \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R}) \\ t &\mapsto \mathrm{SL}_2(\mathbf{Z}) g \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, \quad \text{where } g^{-1}\gamma g = \text{diagonal} \end{aligned}$$

- Linking $\mathrm{Knot}(\gamma)$ and trefoil $\leftrightarrow$ Dedekind–Rademacher cocycle (Ghys)
- Linking $\mathrm{Knot}(\gamma_1)$ and $\mathrm{Knot}(\gamma_2)$ $\leftrightarrow$ Knopp cocycle (Duke–Imamoḡlu–Tóth)

In both cases, one *integrates* cocycles for $\mathrm{SL}_2(\mathbf{Z})$.

Multiplicative cocycles

Consider $\mathbf{C}(z)^\times$:= multiplicative group of non-zero rational functions on $\mathbf{P}^1(\mathbf{C})$.
It is a left $\mathrm{GL}_2(\mathbf{Q})$-module for weight zero action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) \;:=\; f\left(\frac{dz-b}{-cz+a}\right)$$

There is a morphism of $\mathrm{GL}_2(\mathbf{Q})$-modules

$$\mathrm{dlog} \;:\; \mathbf{C}(z)^\times \longrightarrow \mathbf{C}(z)\,;\, f(z) \longmapsto \left(\frac{d}{dz} f(z)\right)\cdot f(z)^{-1}$$

whose kernel is $\mathbf{C}^\times \subset \mathbf{C}(z)^\times$. Note that both p_c and $kn_{c,F}$ are valued in the image of

$$\mathrm{dlog} : \mathbf{C}(z)^\times / \mathbf{C}^\times \hookrightarrow \mathbf{C}(z),$$

and therefore lift formally to multiplicative cocycles (modulo scalars!)

$$\left.\begin{array}{llll} \text{Toy cocycle lift:} & \gamma & \longmapsto & z-(\gamma\infty) \\ \text{Knopp cocycle lift:} & \gamma & \longmapsto & \prod_{Q\sim F} (z-r(Q))^{\mathrm{sgn}\, Q(\gamma\infty)} \end{array}\right\} \in Z^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \frac{\mathbf{C}(z)^\times}{\mathbf{C}^\times}\right)$$

Q : Can we get rid of scalar ambiguity, and lift them to cocycles in $Z^1\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times\right)$?

Lemma: The projection map is an isomorphism:

$$12Z^1_f\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times\right) \xrightarrow{\sim} 12Z^1_{\mathrm{par}}\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times/\mathbf{C}^\times\right).$$

Denote $p^\times$ and $kn_F^\times \in Z^1_f\left(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times\right)$ for the unique lifts of 12th power of p and kn_F.

Values of cocycles

We will now *evaluate* multiplicative cocycles at non-split indefinite forms G.

- Let $G \in \mathcal{F}_D$ with $D > 0$ non-square, then its stabiliser in $\mathrm{SL}_2(\mathbf{Z})$ is generated modulo torsion by the *automorph* of G defined by

  $$\mathrm{Stab}(G) = \pm\langle\gamma_G\rangle \leq \mathrm{SL}_2(\mathbf{Z}), \qquad \gamma_G := \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix}, \quad \begin{array}{c}\text{where } t, u > 0 \text{ min,} \\ \text{such that } t^2 - Du^2 = 4.\end{array}$$

  Computed efficiently using reduction algorithm.
- Given a multiplicative cocycle $\varphi \in Z^1(\mathrm{SL}_2(\mathbf{Z}), \mathbf{C}(z)^\times)$ its *value* is defined by

  $$\varphi[G] := \varphi(\gamma_G)(r(G)) \in \mathbf{P}^1(\mathbf{C}).$$

This value only depends on the $\mathrm{SL}_2(\mathbf{Z})$-orbit of G. For our examples, we have (exercises)

$$\begin{array}{rcl} p^\times[G] & = & \varepsilon_D^{12} \\ kn_F^\times[G] & \in & \mathbf{Q}(\sqrt{D_1}, \sqrt{D_2}) \end{array}$$

Summary. We went through the following motions:

1. Construct ‘interesting’ $\mathrm{SL}_2(\mathbf{Z})$ cocycles valued in the additive group $\mathbf{C}(z)$.
2. Lift them to $\mathrm{SL}_2(\mathbf{Z})$ cocycles valued in the multiplicative group $\mathbf{C}(z)^\times$.
3. Evaluate them at indefinite forms $G \in \mathcal{F}_D$ with $D > 0$ non-square to get *numbers*.

Rigid cocycles

Preview: Will see that specific p-adic limits of the values of multiplicative Knopp cocycles converge to quantities that mimic Gross–Zagier's differences of singular moduli. Replace

$$\begin{array}{rrcl} & \mathrm{SL}_2(\mathbf{Z}) & \circlearrowright & \mathbf{C}(z)^\times \\ \text{by} & \Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p]) & \circlearrowright & \mathcal{M}^\times \end{array}$$

where $\mathcal{M}^\times$:= non-zero meromorphic functions on the p-adic upper half plane $\mathcal{H}_p$.

The p-adic upper half plane $\mathcal{H}_p$

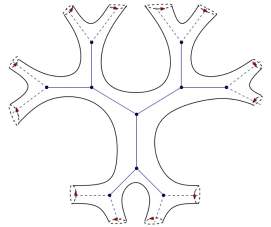
Define p-adic upper half plane $\mathcal{H}_p := \lim_{n\to\infty} \mathcal{H}_p^{\leq n}$ where

$$\mathcal{H}_p^{\leq n} := \left\{ (z_1, z_2) \text{ prim. } : \begin{array}{c} |sz_1 + rz_2|_p \geq p^{-n} \\ \forall (r,s) \in \mathbf{P}^1(\mathbf{Q}) \text{ prim.} \end{array} \right\}$$

$$\subset \mathbf{P}^1_{[z_1:z_2]}(\mathbf{C}_p)$$

It is a rigid analytic space with $\mathcal{H}_p(\mathbf{C}_p) = \mathbf{P}^1(\mathbf{C}_p) \setminus \mathbf{P}^1(\mathbf{Q}_p)$.

Note: Contains $r(G)$ for $G \in \mathcal{F}_D$ with $\left(\frac{D}{p}\right) = -1$.



Picture: Marc Masdeu

$\mathcal{M}$:= uniform limits of rational functions on the affinoid covering $\left\{ \mathcal{H}_p^{\leq n} : n \geq 1 \right\}$.

Rigid cocycles

Go through the same motions as in the case of rational cocycles, inspired by Knopp cocycles.

Let $\Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p])$ and $c = \infty \in \mathbf{P}^1(\mathbf{Q})$. Choose $F \in \mathcal{F}_D$ with $D > 0$ and $\left(\frac{D}{p}\right) = -1$.

1. **Step 1:** Construct interesting additive cocycle
$$\gamma \longmapsto \sum_{Q \in F\cdot\Gamma} \frac{\mathrm{sgn}_{c,Q}(\gamma)}{z - r(Q)} \qquad \in Z^1(\Gamma, \mathcal{M}).$$
2. **Step 2:** Define multiplicative lifts
$$\gamma \longmapsto \prod_{Q \in F\cdot\Gamma} (z - r(Q))^{\mathrm{sgn}_{c,Q}(\gamma)} \qquad \in Z^1(\Gamma, \mathcal{M}^\times / \mathbf{C}_p^\times).$$
Can **never** lift to a Γ-cocycle valued in $\mathcal{M}^\times$. Such is life!
3. **Step 3:** Define the value at $G \in \mathcal{F}_D$ with $D > 0$ and $\left(\frac{D}{p}\right) = -1$ by
    - Restrict multiplicative cocycle (modulo scalars) to $\mathrm{SL}_2(\mathbf{Z})$ and lift its 12th power uniquely to
    $$\Theta_F^\times \in Z^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbf{Z}), \mathcal{M}^\times).$$
    - Evaluate at G using automorph $\gamma_G \in \mathrm{SL}_2(\mathbf{Z})$, setting
    $$\Theta_F^\times[G] := \Theta_F^\times(\gamma_G)(r(G)) \in \mathbf{P}^1(\mathbf{C}_p).$$

(Darmon-V. 2021) algorithm for $\Theta_F^\times[G]$ mod p^m which is polynomial time in precision m.

Plot twist

First ever experiment

When $p = 3$ we compute for

$$\begin{aligned} F &= \langle 1, 1, -1 \rangle \quad \text{disc } 5 \\ G &= \langle 1, 8, -4 \rangle \quad \text{disc } 80 \end{aligned}$$

the RM value of the 3-adic rigid cocycle associated to F at G, and find

$$\Theta_F^{\times}[G] \equiv \frac{24\sqrt{-1} - 7}{25} \pmod{3^{200}}.$$

Next time: Investigate these p-adic invariants experimentally, in the style of Gross–Zagier.