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Rational points on varieties and the Brauer-Manin obstruction

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These notes are in draft form.

Prologue

A central object of interest in arithmetic geometry is the set of k-rational points on a smooth projective geometrically integral k-variety X, denoted $X(k)$. In these notes, we focus on the fundamental problem of determining whether $X(k) \neq \emptyset$.

If you are handed a variety, e.g., $X := V(x^3 + 2y^3 + 10z^3) \subset \mathbb{P}^2$, and asked to determine if it has rational points, a natural first step is to try to find some solutions where the coordinates are small (in other words, solutions of small height). If you are lucky, you can find a point of small height (try to do so with the example above!), and you will thereby have proved that the set $X(\mathbb{Q}) \neq \emptyset$.

In fact, if $X(k) \neq \emptyset$, then an approach like this will give a proof! Any point $P \in X(k)$ has a (finite) height H, so searching in the box $\{x \in \mathbb{P}^n(k) : H(x) \leqslant H\}$ is a finite procedure and will result in you finding the point P.

However, if $X(k) = \emptyset$, then this approach will never terminate. So if your search keeps failing, you cannot tell whether you need to search in a larger box or whether actually there is no rational point. Searching gives no way of certifying that there are no k-points. To certify a lack of points, we need another method. These notes focus on one such method that is known as the Brauer-Manin obstruction.

The goal of these notes There are many excellent references on the Brauer-Manin obstruction (e.g., [6, 15, 17] just to name a few). We do not endeavor to improve on (or replicate!) those references here. Rather these notes should be thought of as a sort of guidebook to the field, giving an overview of the current landscape, with pointers of where to go should you, the reader, decide you wish to explore more. Just as travel guidebooks have particular biases (affordability, nature, good for kids, etc.), so do these notes. They give *my* perspective on how the feedback loop of computation and theory currently manifests in the study of rational points and the Brauer-Manin obstruction.

Notation and conventions Throughout, we will use F to denote an arbitrary field. We write F^{sep} for a fixed separable closure, and $\overline{F}$ for a fixed algebraic closure containing F^{sep}. We use G_F to denote the absolute Galois group $\mathrm{Gal}(F^{sep}/F)$.

We reserve k to denote a number field. We write Ω_k for the set of places of k, and for any $v \in \Omega_k$ we let k_v denote the completion. If $v \in \Omega_k$ is nonarchimedean, then we write $\mathcal{O}_v \subset k_v$ for the valuation ring, $\mathfrak{m}_v$ for the maximal ideal and $\mathbb{F}_v := \mathcal{O}_v/\mathfrak{m}_v$ for the residue field. We write $\mathbb{A}_k := \prod'(k_v, \mathcal{O}_v)$ for the adele ring of k.

We will use X to denote a smooth projective geometrically integral variety over k. For any extension F/k we write X_F for the base change of X to $\operatorname{Spec} F$. We also will use the conventions $\overline{X} := X_{\overline{F}}$ and $X^{\text{sep}} := X_{F^{\text{sep}}}$.

1. Obstructions to the existence of rational points

There is no known general method to determine if the set $X(k)$ is nonempty.[1] Instead, we find a set S that 1) contains $X(k)$ and 2) seems to be computable (at least in some cases). Then, if we can prove that $S = \emptyset$, this will imply that $X(k) = \emptyset$. The set S is typically called an <u>obstruction set</u> since, as we discussed, the emptyness of S *obstructs* the existence of k-points.

1.1. Local obstructions The simplest obstruction set comes from the embeddings of k into one of its completions k_v. Precisely, if v is a place of k, then the embedding $k \hookrightarrow k_v$ induces an inclusion $X(k) \subset X(k_v)$. So if there are no k_v-points on X, then there are also no rational points. In addition, if v is nonarchimedean, then Hensel's lemma implies that there is some n such that $X(k_v) \neq \emptyset$ if and only if $X(\mathcal{O}_v/\mathfrak{m}_v^n) \neq \emptyset$. In particular, it is a finite computation to determine if $X(k_v) \neq \emptyset$ for nonarchimedean places v (and the same is true for archimedean places, see [15, Remark 2.6.4])

Exercise 1.1.1. Show that

$$X := V(x^2 + y^2 + 7z^2) \subset \mathbb{P}^2$$

has no $\mathbb{Q}_7$-points. Conclude that $X(\mathbb{Q}) = \emptyset$.

We may package these obstructions together using the adèles $\mathbb{A}_k$ of k. Just as above, the embedding $k \hookrightarrow \mathbb{A}_k$ allows us to view $X(k)$ as a subset of $X(\mathbb{A}_k)$, and one can prove (see [15, Section 2.6.3]) that

$$X(\mathbb{A}_k) = \prod_v{}'(X(k_v), X(\mathcal{O}_v)).$$

Since, by assumption, X is projective, the valuative criterion for properness implies that the inclusion $X(\mathcal{O}_v) \subset X(k_v)$ is an equality. Hence, $X(A_k) = \prod X(k_v)$.

It turns out that even though $X(\mathbb{A}_k)$ encapsulates the infinitely many completions at once, it is still a finite computation to determine whether $X(\mathbb{A}_k) \neq \emptyset$.

[1] In fact, it may be an undecidable problem! This question of decidability is known as Hilbert's tenth problem; see [14] for more details.

Proposition 1.1.2 (c.f. [15, Thm. 7.7.2]). *Let X be a smooth projective geometrically integral variety over a number field k. Then the set of places where X fails to have k_v-points is finite.*

Proof. Since X is smooth and geometrically integral, then by slicing with sufficiently general hyperplanes, we may obtain a smooth, geometrically integral curve $C \subset X$. By generic smoothness, C necessarily has good reduction away from a finite set of places (see [15, Section 3.2] for more details). Furthermore, the Hasse-Weil bounds [15, Cor. 7.2.1] imply that, for any place of good reduction with $\#\mathbb{F}_v$ sufficiently large, the reduction $C \bmod v$ is guaranteed to have a smooth $\mathbb{F}_v$-point. Hensel's lemma [15, 3.5.63] then implies that C, and hence X, has $\mathbb{Q}_v$ points for such v. Thus we have proved that $X(k_v) \neq \emptyset$ for all v outside the following subset

$$\{v|\infty\} \cup \{C \bmod v \text{ singular}\} \cup \{\#\mathbb{F}_v < N\},$$

where N is some positive integer given by the Hasse-Weil boards. Since this set is finite, we have proved the desired result. ☐

Exercise 1.1.3. *Note:* For this problem, the following specific consequence of Hensel's Lemma will be useful. If p be a prime and $u \in \mathbb{Z}_p^\times$, then

$$u \in \mathbb{Z}_p^{\times 2} \Leftrightarrow \begin{cases} u \bmod p \in \mathbb{F}_p^{\times 2} & \text{if } p \neq 2 \\ u \equiv 1 \pmod 8 & \text{if } p = 2 \end{cases}$$

(1) Let p be an odd prime and let $a, b, c \in \mathbb{Z} - p\mathbb{Z}$. Show that $\{ax^2 : x \in \mathbb{F}_p\}$ and $\{c - by^2 : y \in \mathbb{F}_p\}$ both have cardinality $\frac{p+1}{2}$ and therefore that the sets contain a common value. Use this to show that

$$X := V(ax^2 + by^2 + cz^2) \subset \mathbb{P}^2$$

has a $\mathbb{Q}_p$-point.

(2) Determine whether $X := V(5x^2 + 7y^2 - 3z^2) \subset \mathbb{P}^2$ has $\mathbb{A}_\mathbb{Q}$-points.

(3) Let $a, b, c \in \mathbb{Z}$ be squarefree, pairwise relatively prime integers. Prove that $X := V(ax^2 + by^2 + cz^2)$ has $\mathbb{A}_\mathbb{Q}$-points if and only if a, b, c are not all the same sign and $ax^2 + by^2 + cz^2$ has solutions in $\mathbb{Z}/8abc\mathbb{Z}$ such that for every $p|8abc$, at least two of the coordinates are nonzero modulo p.

1.1.1. The local-to-global principle

Definition 1.1.4. A class of varieties $\mathcal{C}$ satisfies the local-to-global principle (abbreviated LGP) if, for all $X \in \mathcal{C}$,

$$X(\mathbb{A}_k) \neq \emptyset \Leftrightarrow X(k) \neq \emptyset.$$

Examples 1.1.5.

(1) Quadrics, in any number of variables, satisfy the local-to-global principle by the Hasse-Minkowski theorem.

(2) Severi-Brauer varieties, i.e., varieties that are geometrically *isomorphic* to projective space, satisfy the local-to-global principle by the Albert-Brauer-Hasse-Noether theorem.

Note that two classes of varieties given above are all geometrically very close to projective space. That is not just due to limitations in our methods to prove that the loca-to-global principle holds (even though these statements are generally quite difficult to prove). It is because more complicated varieties often can fail the local-to-global principle!

The first counterexample to the local-to-global principle dates back to the 1940's and is due to Lind and Reichardt.

Theorem 1.1.6 ([11,16]). *Let $C \subset \mathbb{P}^3$ be the smooth projective genus 1 curve defined by the two quadrics $2y^2 = w^2 - 17z^2$ and $wz = x^2$. Then*

$$C(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset \quad \text{and} \quad C(\mathbb{Q}) = \emptyset.$$

To prove that C has no rational points despite the presence of the adelic points, we need a *refined* obstruction set, i.e., an intermediate set S that sits between the set of rational points and the set of adelic points.

1.2. An introduction to the Brauer-Manin obstruction The goal of this section is to define the Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$, and show that it is a refined obstruction set, i.e., that we have the following containments:

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k).$$

To do so, we must first introduce the Brauer group.

1.2.1. The Brauer group of a field ([8],[6, Chap. 1],[13, Chap. 4])

Let F be a field.

Definition 1.2.1.

(1) A central simple algebra over F (abbreviated CSA/F) is a finite dimensional F-algebra $\mathcal{A}$ whose center is exactly F and that has no nontrivial two-sided ideals.

(2) Two central simple algebras $\mathcal{A}, \mathcal{B}$ over F are said to be Brauer equivalent if there exist positive integers n, m such that $\mathcal{A} \otimes M_m(F) \simeq \mathcal{B} \otimes M_n(F)$.

Theorem 1.2.2 ([8, Prop. 2.4.8 and Thm. 4.4.7]). *Let F be a field. The set of Brauer equivalences classes of central simple algebras over F forms a torsion abelian group under tensor product, where the identity element is the class of $M_n(F)$; this group is isomorphic to $H^2(G_F, (F^{\mathrm{sep}})^{\times})$.*

Definition 1.2.3. The group from Theorem 1.2.2 is known as the Brauer group of F and is denoted $\mathrm{Br}\, F$.

Example 1.2.4. Assume that F has characteristic different from 2 and let $a, b \in F^{\times}$. Then the quaternion algebra

$$\mathcal{A}_{a,b} := F \otimes F \cdot i \otimes F \cdot j \otimes F \cdot ij, \quad i^2 = a, j^2 = b, ji = -ij$$

is a CSA/F. It has order dividing 2 in $\mathrm{Br} F$ [8, Cor. 1.5.3].

Exercise 1.2.5. Let F be a field of characteristic different from 2 and let $a, b \in F^\times$.

Prove that $\mathcal{A}_{a,b} \simeq M_2(F)$ if and only if there is some $x, y, z \in F$, not all zero, such that $ax^2 + by^2 = z^2$. Conclude that $(a, b) := [\mathcal{A}_{a,b}] \in \mathrm{Br}\, F$ is trivial if and only if the conic $C_{a,b} : ax^2 + by^2 = z^2$ has an F-rational point, and that $C_{a,b}(F) \neq \emptyset$ if and only if $a \in N(k(\sqrt{b})^\times)$. (By symmetry this is equivalent to $b \in N(k(\sqrt{a})^\times)$). *Note:* One can also show that $\mathcal{A}_{a,b} \oplus_k \mathcal{A}_{a,c} \simeq M_2(\mathcal{A}_{a,bc})$, (see [8, Lemma 1.5.2]) which implies that in $\mathrm{Br}\, F$, we have $(a, b)(a, c) = (a, bc)$.

The above example is a special case of a general correspondence. To any central simple algebra over F, one can associate a Severi-Brauer variety, and vice versa [8, Section 5.2].

Our interest in the Brauer group stems from the fact that, over global fields, it encodes the abelian reciprocity laws. This is encapsulated by the fundamental exact sequence of global class field theory.

Theorem 1.2.6 ([15, Thms. 1.5.34 and 1.5.36])**.** *Let k be a number field. For each place v, there is an injective homomorphism*

$$\mathrm{inv}_v \colon \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z},$$

that is an isomorphism for nonarchimedean v. For archimedean v, the image is $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ if $k_v = \mathbb{R}$ and 0 if $k_v = \mathbb{C}$. Furthermore, these isomorphisms fit together in the following short exact sequence

$$0 \to \mathrm{Br}\, k \to \oplus_v \mathrm{Br}\, k_v \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0. \tag{1.2.7}$$

Remark 1.2.7.1. When working with explicit computations, one has to take care that the maps inv_v are defined in a globally compatible way. An arbitrary collection of isomorphisms $\phi_v \colon \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z}$ for nonarchimedean v will not necessarily give an exact sequence as in (1.2.7). See [6, Definition 13.1.7] for a definition of inv_v. The explicit examples in these notes will restrict to 2-torsion elements to avoid this subtlety.

Example 1.2.8. Let $a, b \in k^\times$ and let v be a place of k. Using Example 1.2.4, one can check that $\mathcal{A}_{a,b} \in \mathrm{Br}\, k_v$ is nontrivial if and only if the Hilbert symbol $(a, b)_v = -1$. Thus, the fact $\sum_v \mathrm{inv}_v(\mathcal{A}_{a,b}) = 0$ is exactly the product law

$$\prod_v (a, b)_v = 1,$$

which is also equivalent to quadratic reciprocity.

1.2.2. The Brauer group of a variety The notion of the Brauer group of a field can be generalized to the Brauer group of a scheme using étale cohomology. Namely, we can define $\mathrm{Br}\, X := H^2_{et}(X, \mathbb{G}_m)$.[2]

For our purposes, the following properties will be particularly useful.

[2]There are several ways to extend the definition of the Brauer group from a field to a scheme. This definition is sometimes called the <u>cohomological Brauer group</u>, since, in general, it is not necessarily

Theorem 1.2.9.

(1) *For any field* F *we have* $\operatorname{Br} F = \operatorname{Br} \operatorname{Spec} F$.

(2) *The Brauer group is functorial, i.e., if there is a morphism of varieties* $f\colon X \to Y$, *then we have a homomorphism* $f^*\colon \operatorname{Br} Y \to \operatorname{Br} X$.

(3) *If* X *is a smooth geometrically integral variety over a field* F *of characteristic* 0, *then the inclusion of the generic point* $\iota\colon \operatorname{Spec} \mathbf{k}(X) \to X$ *gives an injective homomorphism* $\iota^*\colon \operatorname{Br} X \hookrightarrow \operatorname{Br} \mathbf{k}(X)$.[6, Thm. 3.5.5]

(4) *The Brauer group is a birational invariant of smooth projective varieties, i.e., if* $f\colon X \dashrightarrow Y$ *is a birational map between two smooth projective varieties, then* $f^*\colon \operatorname{Br} \mathbf{k}(Y) \to \operatorname{Br} \mathbf{k}(X)$ *sends* $\operatorname{Br} Y$ *isomorphically to* $\operatorname{Br} X$. [6, Cor. 6.2.11]

(5) *If* X *is a smooth projective variety and let* $U \subset X$ *be an open subset such that* $\operatorname{codim}(X \setminus U) \geqslant 2$, *then* $\operatorname{Br} X = \operatorname{Br} U$. [6, Thm. 3.7.6]

1.2.3. The Brauer-Manin set In this section, we will show how the Brauer group of a variety X carves out a subset of $X(\mathbb{A}_k)$ that contains the k-rational points. This subset was introduced by Manin [12] and is known as the Brauer-Manin set.

For any extension F/k, the functoriality of the Brauer group (Theorem 1.2.9(2)) gives a pairing

$$\operatorname{Br} X \times X(F) \to \operatorname{Br} F, (\alpha, x) \mapsto x^*\alpha := \langle \alpha, x \rangle,$$

where we view the point $x \in X(F)$ as a map $x\colon \operatorname{Spec} F \to X$.

If X is projective, then $X(\mathbb{A}_k) = \prod_v X(k_v)$ and so we may apply these pairings componentwise to obtain a pairing

$$\operatorname{Br} X \times X(\mathbb{A}_k) \to \prod_v \operatorname{Br} k_v.$$

Using integral models and properties of Brauer groups of local fields, one can show that the image of this pairing actually lands in $\oplus_v \operatorname{Br} k_v$, i.e., that for every $\alpha \in \operatorname{Br} X$ there exists a finite set $S_\alpha \subset \Omega_k$ such that $\langle \alpha, - \rangle\colon X(k_v) \to \operatorname{Br} k_v$ is identically 0 for all $v \notin S_\alpha$[15, Prop. 8.2.1]. Therefore, we have the following commutative diagram.

$$\begin{array}{ccc} \operatorname{Br} X \times X(k) & \hookrightarrow & \operatorname{Br} X \times X(\mathbb{A}_k) \\ \downarrow & & \downarrow \\ \operatorname{Br} k & \longrightarrow & \oplus_v \operatorname{Br} k_v \end{array}$$

Recall from Theorem 1.2.6, the bottom vertical arrow fits into an exact sequence, so we may extend our diagram as follows.

$$\begin{array}{ccccccccc} & & \operatorname{Br} X \times X(k) & \hookrightarrow & \operatorname{Br} X \times X(\mathbb{A}_k) & \overset{\phi}{\searrow} & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \operatorname{Br} k & \longrightarrow & \oplus_v \operatorname{Br} k_v & \xrightarrow{\sum \operatorname{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

equal to other generalizations. However, in the context that we will work in (smooth varieties over a field of characteristic 0), all of the generalization coincide, so we will simply refer to this as the Brauer group.

Observe that since the bottom row is a complex and the diagram commutes, then the induced map $\operatorname{Br} X \times X(k) \to \mathbb{Q}/\mathbb{Z}$ is identically 0. In particular, for all $\alpha \in \operatorname{Br} X$, the set $X(k)$ is contained in the set of adelic points orthogonal to α, i.e.,

$$X(k) \subset X(\mathbb{A}_k)^{\alpha} := \{(x_v) \in X(\mathbb{A}_k) : \langle \alpha, (x_v) \rangle = 0\}.$$

Taking the intersection $X(\mathbb{A}_k)^{\alpha}$ for all α, we obtain the Brauer-Manin set

$$X(\mathbb{A}_k)^{\operatorname{Br}} := \bigcap_{\alpha \in \operatorname{Br} X} X(\mathbb{A}_k)^{\alpha} = \{(x_v) \in X(\mathbb{A}_k) : \langle \alpha, (x_v) \rangle = 0 \, \forall \alpha \in \operatorname{Br} X\}. \tag{1.2.10}$$

Exercise 1.2.11. Let X be a smooth projective geometrically integral variety over a number field k and let π denote the structure morphism $\pi \colon X \to \operatorname{Spec} k$.

(1) Let $\alpha_0 \in \operatorname{Br} k$. Show that $X(\mathbb{A}_k)^{\pi^* \alpha_0} = X(\mathbb{A}_k)$.

(2) Let $\alpha, \beta \in \operatorname{Br} X$. Show that

$$X(\mathbb{A}_k)^{\alpha} \cap X(\mathbb{A}_k)^{\beta} = \bigcap_{\gamma \in \langle \alpha, \beta \rangle} X(\mathbb{A}_k)^{\gamma}$$

These exercises show that the Brauer-Manin set depends only on the quotient $\operatorname{Br} X / \operatorname{im}(\pi^* \colon \operatorname{Br} k \to \operatorname{Br} X)$. We write $\operatorname{Br}_0 X := \operatorname{im}(\pi^* \colon \operatorname{Br} k \to \operatorname{Br} X)$ and refer to elements in $\operatorname{Br}_0 X$ as constant Brauer classes.

2. Computing the Brauer-Manin obstruction

Now we have successfully defined a refined obstruction set, the Brauer-Manin set

$$X(k) \subset X(\mathbb{A}_k)^{\operatorname{Br}} \subset X(\mathbb{A}_k),$$

but we have yet to see whether this is *useful*. Utility has a theoretical component and a practical or computational component.

(1) [Theory] Do there exist varieties where $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$ yet $X(\mathbb{A}_k) \neq \emptyset$?
(2) [Practice/Computation] Can we compute the Brauer-Manin set?

The answer to the first question is a resounding *YES!* In fact, there are many such examples. When Manin introduced this obstruction, he showed that several of the known failures of the local-to-global principle could be explained by the Brauer-Manin obstruction. Since then several more examples have been constructed, which together lead to the expectation that if a class of variety can have nontrivial Brauer group, it is likely that there exists such a variety with a Brauer-Manin obstruction to the existence of rational points.[3]

The answer to the second question is more mixed. On the one hand, the Brauer-Manin set *has* been computed in several examples, and for some classes of varieties, it is more or less standard to do so (e.g., conic bundles over $\mathbb{P}^1$ [6, Prop. 11.3.4]). On the other hand, there is no general effectivity result for the Brauer-Manin set, nor is there an approach that is known to work in full generality.

[3] Unless there is an "obvious" reason why not, e.g., the varieties always have a rational point, like in the case of del Pezzo surfaces of degree 1.

The best effectivity results to date are due to Kresch and Tschinkel [9, 10]. They prove that if $\operatorname{Pic} \overline{X}$ is torsion free, then the algebraic Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}_1}$ is effectively computable. The algebraic Brauer-Manin set is, by definition, the set of adelic points orthogonal to those Brauer classes that become trivial after passage to the algebraic closure; this is often larger than the Brauer-Manin set. Kresch and Tschinkel can improve their results for surfaces. In that case (and still under the assumption that $\operatorname{Pic} \overline{X}$ is torsion-free), they prove that for any positive integer n, there is an effectively computable set X_n such that

$$X(\mathbb{A}_k)^{\mathrm{Br}} \subset X_n \subset X(\mathbb{A}_k)^{\mathrm{Br}\, X[n]}.$$

In particular, if there exists an effective bound on the exponent of $\operatorname{Br} X / \operatorname{Br}_0 X$, then this would imply that the Brauer-Manin set is effectively computable for surfaces with torsion-free geometric Picard group.

Despite the lack of general tools for computing the Brauer-Manin obstruction, there is a general framework that is often helpful in computing the group structure of $\operatorname{Br} X / \operatorname{Br}_0 X$ as an abstract torsion abelian group.

2.1. The Hochschild-Serre spectral sequence and a filtration of the Brauer group We will leverage the Hochschild-Serre spectral sequence in étale cohomology (applied to the Galois cover $\overline{X} \to X$ and the sheaf $\mathbb{G}_m$):

$$\text{(2.1.1)} \qquad H^p\left(G_k, H^q_{et}(\overline{X}, \mathbb{G}_m)\right) \Rightarrow H^{p+q}_{et}(X, \mathbb{G}_m).$$

to give a filtration of the Brauer group. The exact sequence of low degree terms (see [15, Prop. 6.7.1]) is

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{G_k} \to \operatorname{Br} k \to \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right) \to H^1(G_k, \operatorname{Pic} \overline{X}) \to 0;$$

(here we use the assumption that k is a number field and so $H^3(G_k, \overline{k}^{\times}) = 0$ [15, Remark 6.7.10] to obtain the rightmost 0). Using the definition of $\operatorname{Br}_0 X$, we obtain a short exact sequence

$$\text{(2.1.2)} \qquad 0 \to \operatorname{Br}_0 X \to \operatorname{Br}_1 X := \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right) \to H^1(G_k, \operatorname{Pic} \overline{X}) \to 0.$$

The subgroup $\mathrm{Br}_1 X$ is called the algebraic Brauer group of X, and the quotient $\operatorname{Br} X / \operatorname{Br}_X$ is called the transcendental Brauer group of X.

The transcendental Brauer group can also be studied using the Hochschild-Serre spectral sequence. Indeed, the higher degree terms yield the following exact sequence (again using the assumption that k is a number field and so $H^3(G_k, \overline{k}^{\times}) = 0$) [6, 5.24]

$$\text{(2.1.3)} \qquad 0 \to \frac{\operatorname{Br} X}{\operatorname{Br}_1 X} \to \left(\operatorname{Br} \overline{X}\right)^{G_k} \to H^2(G_k, \operatorname{Pic} \overline{X}).$$

Thus, if we have a good enough understanding of $\operatorname{Pic} \overline{X}$ and $\operatorname{Br} \overline{X}$ as Galois modules, we can leverage (2.1.2) and (2.1.3) to compute the Brauer group modulo constants. (See [6, Section 5.4] for more details on these approaches.)

2.2. Extended example of computing the Brauer-Manin obstruction Let $X \subset \mathbb{P}^4$ be given by the vanishing of the following two quadrics

$$st - x^2 + 5y^2, \quad (s+t)(s+2t) - x^2 + 5z^2.$$

This variety was first studied by Birch and Swinnerton-Dyer [1].

(1) Note that an intersection of quadrics in $\mathbb{P}^3$ is a genus 1 curve, and any smooth genus 1 curve over a finite field $\mathbb{F}$ has an $\mathbb{F}$-point. Prove that $X \cap V(z)$ is smooth modulo p for all $p \neq 2, 5$. Use this to prove that $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 2, 5$.

(2) Building on the previous part, show that $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$.

(3) Using Exercise 1.2.5, show that $(5, \frac{s}{t})$ and $(5, \frac{s+t}{s+2t})$ are trivial in $\operatorname{Br} \mathbf{k}(X)$.

(4) Using the previous part and the remark at the end of Exercise 1.2.5 show that, in $\operatorname{Br} \mathbf{k}(X)$

$$\mathcal{A} := \left(5, \frac{s+t}{s}\right) = \left(5, \frac{s+2t}{s}\right) = \left(5, \frac{s+t}{t}\right) = \left(5, \frac{s+2t}{t}\right).$$

Additionally show that for every point $P \in X - V(s,t)$, there is an open set $P \in U_P \subset X - V(s,t)$ such that at least one of $\frac{s+t}{s}, \frac{s+t}{t}, \frac{s+2t}{s}, \frac{s+2t}{t}$ is regular and invertible on U.
Note: Since $V(s,t) \subset X$ is codimension 2 in X, this together with the purity theorem (see [15, Thm. 6.8.3]) allows us to conclude that $\mathcal{A} \in \operatorname{Br} X$.

(5) Show that $X(\mathbb{A}_\mathbb{Q})^\mathcal{A} = \emptyset$ and hence $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$. (*Sketch*: First show that for all $p \neq 5$ and $P_p \in X(\mathbb{Q}_p)$, at least one of $\frac{s+t}{s}, \frac{s+t}{t}, \frac{s+2t}{s}, \frac{s+2t}{t}$ is a p-adic unit at P_p. Then, noting that $\mathbb{Q}_p(\sqrt{5})/\mathbb{Q}_p$ is unramified for $p \neq 5$, use Problem (1.2.5) to deduce that $\mathcal{A}(P_p) = 0 \in \operatorname{Br} \mathbb{Q}_p$. Lastly, show that $\mathcal{A}(P_5) \neq 0 \in \operatorname{Br} \mathbb{Q}_5$ for all $P_5 \in X(\mathbb{Q}_5)$.)

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