Rational points on varieties and the Brauer-Manin obstruction

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My work appearing in this talk was predominantly completed on the lands of the Coast Salish, Duwamish, Stillaguamish, and Suquamish nations, & I am reporting on it from the lands of the East Shoshone and Ute nations.

From Lecture 1

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(From Lecture 2)

Brauer classes **want** to obstruct adelic points

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How does this principle translate in the context of varying extensions?

Can we leverage what we know about $X(\mathbb{A}_k)^{\mathrm{Br}}$ to study $\{X(\mathbb{A}_L)^{\mathrm{Br}} : L/\mathbb{Q}\}$?

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[Liang '22 + CTSSD '87] For all number fields k, $\exists$ Châtelet surface V/k & quadratic ext'n L/k s.t. $\overline{V(k)} = V(\mathbb{A}_k)^{\mathrm{Br}} = V(\mathbb{A}_k)$ but $V(\mathbb{A}_L)^{\mathrm{Br}} \subsetneq V(\mathbb{A}_k)$.

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"Almost always"

$\exists\, L/k$ such that $X(\mathbb{A}_L)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_L)$.

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[Nakahara, Roven] $X \leftrightsquigarrow V(y^2 - az^2 = p(t)x^2)$

If $\exists v \in \Omega_F$ s.t. $v(a) < 0$ or

F splitting field of $p(t)$

$a \notin F_v^{\times 2}$ & X_F has bad red. mod v

Then $\exists\, L/k$ such that $X(\mathbb{A}_L)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_L)$.

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Need to ask for less control

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<u>Lemma</u> [obs. by Wittenberg; in Creutz, Viray]

Given X/k, K/k ext'n of # fields, $B \subset \mathrm{Br}\, X_K$.

Then $X(\mathbb{A}_k)^{\mathrm{Cor}(B)} \subset X(\mathbb{A}_K)^B$.

Can we leverage what we know about $X(\mathbb{A}_k)^{\mathrm{Br}}$ to study $\{L/\mathbb{Q} : X(\mathbb{A}_L)^{\mathrm{Br}} \neq \varnothing\}$?

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Given X/k, K/k ext'n of # fields, $B \subset \operatorname{Br} X_K$.

Then $X(\mathbb{A}_k)^{\mathrm{Cor}(B)} \subset X(\mathbb{A}_K)^{B}$. In particular,

1. $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing \Rightarrow X(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$, and
2. $X(\mathbb{A}_k) \subset X(\mathbb{A}_K)^{\mathrm{Res}(\operatorname{Br} X_k[[K:k]])}$

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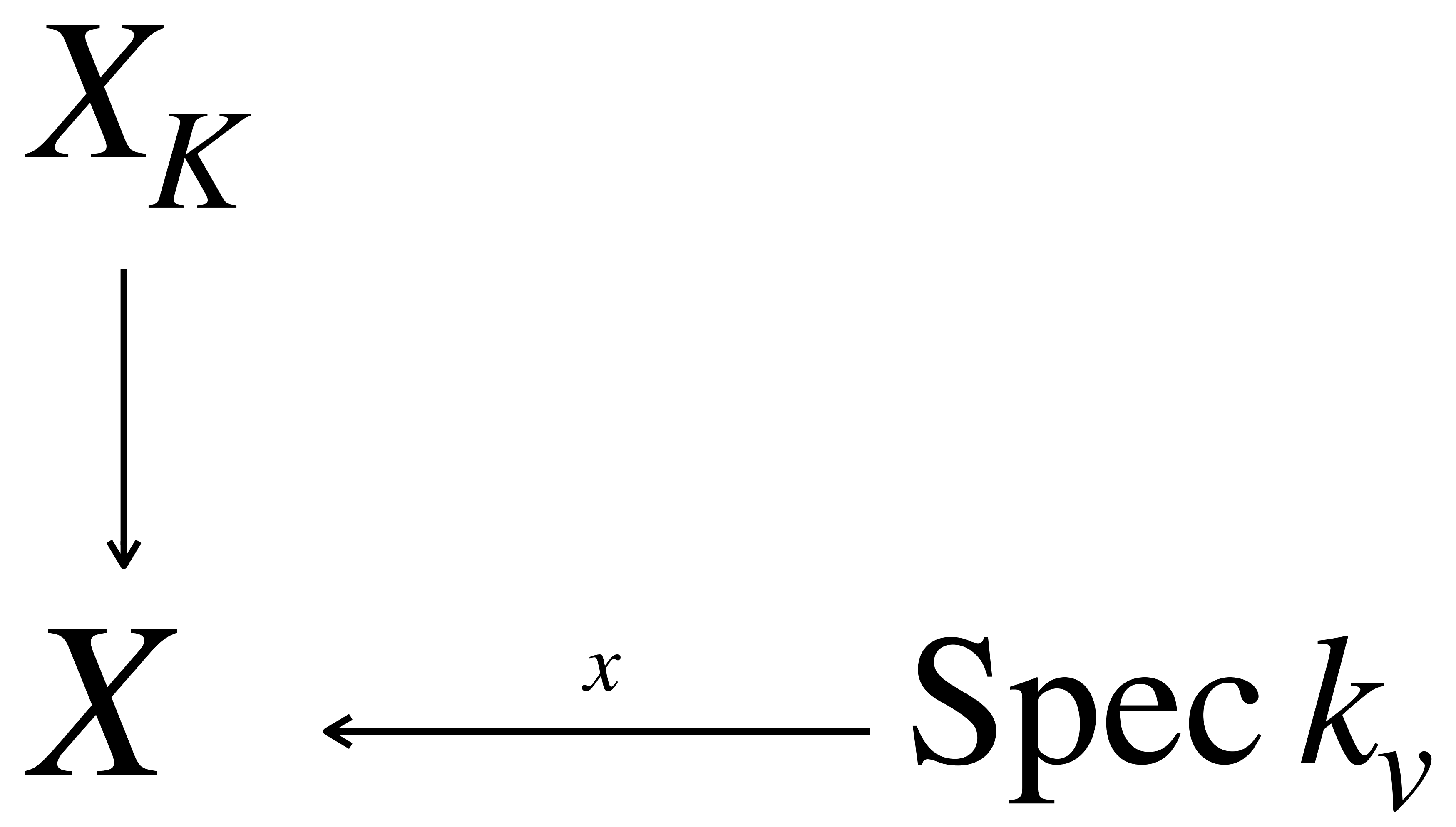

Lemma $X(\mathbb{A}_k)^{\mathrm{Cor}(B)} \subset X(\mathbb{A}_K)^{B}$

Let $\alpha \in \operatorname{Br} X_K$.

$$\begin{array}{ccc} X_K & \xleftarrow{\;y\;} & \operatorname{Spec} K \otimes k_v \\ \downarrow & & \downarrow \\ X & \xleftarrow{\;x\;} & \operatorname{Spec} k_v \end{array}$$

<u>Lemma</u> $X(\mathbb{A}_k)^{\mathrm{Cor}(B)} \subset X(\mathbb{A}_K)^{B}$

Let $\alpha \in \mathrm{Br}\, X_K$.

Then $\mathrm{Cor}(y^*\alpha) = x^*\mathrm{Cor}(\alpha)$.

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$$\Rightarrow \sum_v \mathrm{inv}_v(x_v^*\mathrm{Cor}(\alpha)) = \sum_v \mathrm{inv}_v \mathrm{Cor}(y^*\alpha) = \sum_v \sum_w \mathrm{inv}_w y_w^* \alpha.$$

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Corollary If $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$, then

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<u>Lemma</u> .... $X(\mathbb{A}_k) \subset X(\mathbb{A}_K)^{\mathrm{Res}(\mathrm{Br}\, X_k[[K:k]])}$

If $X(\mathbb{A}_k) \neq \varnothing$ & $\mathrm{Res}(\mathrm{Br}\, X_k[[K:k]])$ captures the Brauer-Manin obstruction, then $X(\mathbb{A}_K)^{\mathrm{Br}} \neq \varnothing$.

Over which extensions *must* the Brauer-Manin obstruction **vanish**?

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Example Corollary:

Let $\pi : X \to \mathbb{P}^n_k$ be a locally soluble conic bundle. Then $\exists K/k$ finite s.t. $\forall$ even degree L/k linearly disjoint from K,

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Example Corollary:

Let X be a locally soluble cubic surface. Then $\exists K/k$ finite s.t. $\forall L/k$ linearly disjoint from K & $3 \mid [L : k]$,

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Example Corollary:

Let X be a locally soluble quartic del Pezzo. Then $\exists K/k$ finite s.t. $\forall\, L/k$ linearly disjoint from K & $2 \mid [L:k]$, $X(\mathbb{A}_L)^{\mathrm{Br}} \neq \emptyset$.

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Let X be a locally soluble [type of variety]. Then $\exists\, K/k$ finite s.t. $\forall\, L/k$ linearly disjoint from K & $M \mid [L:k]$,

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Are these results surprising?

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Let $\pi\colon X \to \mathbb{P}^n_k$ be a locally soluble conic bundle.

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Let $t \in \mathbb{P}^n(k)$. Then X_t is a conic over k.

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Let $\pi\colon X \to \mathbb{P}^n_k$ be a locally soluble conic bundle.
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Let $t \in \mathbb{P}^n(k)$. Then X_t is a conic over k.

If $X(\mathbb{A}_k) \neq \emptyset$, then can find points over even degree extensions approximating any finite set of local conditions.

Example Corollary:

Let $\pi : X \to \mathbb{P}^n_k$ be a locally soluble quartic del Pezzo. Then $\exists\, K/k$ finite s.t. $\forall\, L/k$ linearly disjoint from K & $2 \mid [L : k]$,

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In general, **no** construction of quadratic points.

However, if $X(\mathbb{A}_k) \neq \emptyset$, then, over an odd deg ext'n K/k, X_K is birational to a double cover of $\mathbb{P}^2$.

Over which extensions *must* the Brauer-Manin obstruction **vanish**?

$$X(\mathbb{A}_k) \neq \emptyset \;\&\; X(\mathbb{A}_K)^{\mathrm{Br}} = X(\mathbb{A}_K)^{\mathrm{Res}(\mathrm{Br}\, X_k[[K:k]])} \Rightarrow X(\mathbb{A}_K)^{\mathrm{Br}} \neq \emptyset$$

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(Assume we solved this, at least over k)

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Thm [Roven]

Let $\pi : X \to \mathbb{P}^1$ be a conic bundle. Then

$\forall L/k$ quadratic

$$X(\mathbb{A}_L)^{\mathrm{Res}_{L/k}(\mathrm{Br}\,\mathrm{X})} \neq \varnothing \Leftrightarrow X(\mathbb{A}_L) \neq \varnothing.$$

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Thm [Roven]

Let $\pi\colon X \to \mathbb{P}^1$ be a Châtelet surface. Then

$\forall L/k$ even degree

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Let $\pi\colon X \to \mathbb{P}^1$ be a conic bundle. Then $\exists K/k$ finite s.t. $\forall L/k$ quadratic and linearly disjoint from K,

$$X(\mathbb{A}_L)^{\mathrm{Br}} \neq \varnothing \Leftrightarrow X(\mathbb{A}_L) \neq \varnothing.$$

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$$X(L) \neq \varnothing \Leftrightarrow X(\mathbb{A}_L) \neq \varnothing.$$

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Thm [Creutz, Viray]
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1. $\forall v$, there exists quadratic L_w/k_v s.t. $X(L_w) \neq \emptyset$.
2. If all rank 4 quadrics containing X are defined $/k$, then $\exists$ quadratic L/k s.t. $X(\mathbb{A}_L)^{\mathrm{Br}} \neq \emptyset$.

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Let X/k be a quartic del Pezzo. Then

1. $\forall v$, there exists quadratic L_w/k_v s.t. $X(L_w) \neq \emptyset$.
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Does there **always** exist quadratic L/k with $X(L) \neq \emptyset$?

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Does there **always** exist quadratic L/k with $X(L) \neq \emptyset$?

Does there exist a family of varieties where $\mathrm{Br}/\mathrm{Br}_0$ is 2-torsion but there is a member with **no** quadratic points?

Case of interest: $X(\mathbb{A}_k)^{\mathrm{Br}} = \varnothing$

Over which extensions *must* the Brauer-Manin obstruction **persist**?

Over which extensions *must* the Brauer-Manin obstruction **vanish**?

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I.e., when does $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset \Rightarrow X(\mathbb{A}_L)^{\mathrm{Br}} = \emptyset$?

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I.e., when does $X(\mathbb{A}_k)^{\mathrm{Br}} = \varnothing \Rightarrow X(\mathbb{A}_L)^{\mathrm{Br}} = \varnothing$?

Example: $Q \subset \mathbb{P}^n$ quadric, $\mathrm{Br}\, Q = \mathrm{Br}_0\, Q$,

Springer's theorem:

$$Q(F) = \varnothing \Rightarrow Q(F') = \varnothing \ \forall F'/F \text{ odd degree}$$

Over which extensions *must* the Brauer-Manin obstruction **persist**?

Example: $X = V(Q_0, Q_\infty) \subset \mathbb{P}^4$ quartic del Pezzo

Springer's theorem + Amer, Brumer theorem:

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Does $X(\mathbb{A}_k)^{\mathrm{Br}} = \varnothing \Rightarrow X(\mathbb{A}_L)^{\mathrm{Br}} = \varnothing$ for L/k odd?

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Yes! By [Colliot-Thélène, Coray '79]+ [Swinnerton-Dyer '99]

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Example: X cubic surface

Cassels, Swinnerton-Dyer **conjecture**

$$X(F) = \emptyset \Rightarrow X(F') = \emptyset \; \forall F'/F \text{ with } 3 \nmid [F' : F].$$

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Cassels, Swinnerton-Dyer **conjecture**

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