Quasi local quantities and angular momentum in general relativity

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Glimpses of Mathematics, Now and Then:
A Celebration of Karen Uhlenbeck's 80th birthday

Happy birthday , Karen!
For our friendship of fifty years .

Overview

- This talk is about mathematical general relativity and is based on joint work with Po-Ning Chen, Jordan Keller, Mu-Tao Wang and Ye-Kai Wang.
- Angular momentum at null infinity measures the angular momentum carried away by the gravitational waves.
- The "supertranslation ambiguity" causes major difficulty to properly define angular momentum at null infinity.
- We will discuss how to resolve this using the theory of quasilocal conserved quantities.

Motivation for Quasilocal Mass and Angular Momentum

- A fundamental difficulty with a local notion of mass in general relativity is that there is no density for gravitation, by Einstein's equivalence principle. Such a density would depend upon the first order differentiation of the spacetime metric, but a choice of normal coordinates renders such a measurement trivial at a chosen point.
- Instead try to associate convserved quantities to a spacetime region enclosed by a two-surface. The importance of such a notion was clear to Penrose, who listed the search for a definition of quasilocal mass and a definition of quasilocal angular momentum as his number one and two problems in classical general relativity (Penrose, *Some unsolved problems in classical general relativity*, Seminar on Differential Geometry (1982)).

Motivation for Quasilocal Mass, Ctd.

- There are many reasons to search for such a concept. Many important statements in general relativity make sense only with a good definition and understanding of quasilocal mass. For example, it allows us to discuss the gravitational binding energy of two bodies rotating around one another.
- In addition, a good definition of quasilocal mass should help to control the dynamics of the gravitational field. Such control would complement that of the energy method in hyperbolic equations, which encounters serious difficulties in the stability analysis of the Kerr metric.

Description of Wang-Yau Quasilocal Mass

- We associate a quasilocal mass to a two-dimensional spacetime surface which is a topological two-sphere (the boundary of a three-dimensional spacetime region).
- The Wang-Yau quasilocal mass depends upon both the extrinsic and intrinsic geometry of the surface.
- We suppose the surface is spacelike, i.e. the induced metric σ is Riemannian.
- In addition, we assume that the surface has spacelike mean curvature vector field H, such that $|H| > 0$.

Description of Wang-Yau Quasilocal Mass, Ctd.

- We define the connection one-form of the surface's normal bundle by
$$\alpha_H(\cdot) := g\Big(\nabla^N_{(\cdot)} \frac{J}{|H|}, \frac{H}{|H|}\Big),$$
where (N, g) denotes the ambient spacetime and J is the future timelike vector given by reflection of H through the incoming light cone.
- (Wang-Y., 2009) To evaluate the quasilocal mass of a 2-surface Σ with the physical data (σ, H, α_H), one solves the optimal isometric embedding equation, which gives an embedding of Σ into the Minkowski spacetime with the image surface Σ_0 that has the same induced metric as Σ, i.e. σ.
- One then compares the extrinsic geometries of Σ and Σ_0 and evaluates the quasilocal mass from $\sigma, |H|, |H_0|, \alpha_H$, and α_{H_0}.

Definition of Wang-Yau Quasilocal Mass

- Consider a physical surface Σ with physical data $(\sigma, |H|, \alpha_H)$.
- Given an isometric embedding $X : \Sigma \to \mathbb{R}^{3,1}$ of σ, let Σ_0 be the image $X(\Sigma)$ and $(\sigma, |H_0|, \alpha_{H_0})$ be the data of Σ_0.
- Let T be a future timelike unit Killing field of $\mathbb{R}^{3,1}$ and define $\tau = -\langle X, T \rangle$.

Definition of Wang-Yau Quasilocal Mass, Ctd.

- Define a function ρ and a 1-form j_a on Σ:

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}}$$

$$j_a = \rho\nabla_a\tau - \nabla_a\left(\sinh^{-1}\left(\frac{\rho\Delta\tau}{|H_0||H|}\right)\right) - (\alpha_{H_0})_a + (\alpha_H)_a.$$

- The optimal isometric embedding equation is

$$\begin{cases} \langle dX, dX\rangle & = \sigma \\ \nabla^a j_a & = 0 \end{cases}$$

- The quasilocal mass is then

$$E(\Sigma, X, T) = \frac{1}{8\pi}\int_\Sigma \rho.$$

Non-negativity and Rigidity

- A prototype form of the quasilocal mass (due to Brown-York, Liu-Yau, Booth-Mann, Kijowski, etc) is

$$\frac{1}{8\pi}\int_\Sigma |H_0| - \frac{1}{8\pi}\int_\Sigma |H|$$

  The positivity is proved by Shi-Tam and Liu-Yau.

- However, for a surface in the Minkowski spacetime, the above expression may not be zero.
- The optimal isometric embedding equation gives the necessary correction, so the definition (W.-Yau) is positive in general, and zero for surfaces in the Minkowski spacetime.
- The embedded surface Σ_0 is the "unique" surface in the Minkowski spacetime that best matches the physical surface Σ. If the original surface Σ happens to be a surface in the Minkowski spacetime, the above procedure identifies $\Sigma_0 = \Sigma$ up to a global isometry.

Limiting Properties

- In general, the optimal isometric embedding equation is difficult to solve. However, in a perturbative configuration, when a family of surfaces limit to a surface in the Minkowski spacetime, the optimal isometric embedding equation is solvable, subject to the positivity of the limiting mass.
- One way to test and apply the Wang-Yau definition is to consider its limit at spatial and null infinity, in hopes of recovering the ADM mass and Bondi-Trautman mass, respectively.

Large Sphere Limit at Spatial Infinity

- We consider an initial data set (M, g, k) which is asymptotically flat; that is, there is a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to a union of complements of balls in $\mathbb{R}^3$.
- Assuming $g - \delta = O(r^{-p})$ and $k = O(r^{-q})$ as $r \to \infty$, where $p > \frac{1}{2}$ and $q > \frac{3}{2}$, the ADM mass

$$\frac{1}{16\pi} \int_{S^2_\infty} \sum_{i,j} (g_{ij,j} - g_{jj,i}) \nu^i$$

is well-defined (Bartnik) and the positive mass theorem (Schoen-Yau, Witten) holds.

Large Sphere Limit at Spatial Infinity, Ctd.

- The limit of our quasilocal local mass calculated on the coordinate spheres S_r recovers the ADM definitions under the classical asymptotics ($p > \frac{1}{2}$ and $q > \frac{3}{2}$).
- For more general asymptotics (for example, $p \leq \frac{1}{2}$ or $q \leq \frac{3}{2}$) the ADM definitions are longer valid. In these cases, our definition gives a well-defined conserved quantity that satisfies desirable rigidity and invariance properties.
- The wilder asymptotic behavior of the physical data is captured by the optimal isometric embedding equation and is countered by the reference data.

Description of Null Infinity

- An idealized distant observer is situated at future null infinity $\mathscr{I}^+$, where light rays approach along null geodesics.
- Descriptions of $\mathscr{I}^+$ of an isolated gravitating system include:
  - Bondi-Sachs coordinates (Bondi et al. 1962, Sachs 1962)
  - Penrose conformal compactification (Penrose 1965)
  - Christodoulou-Klainerman (1993)
- Bondi-Sachs and Penrose formalisms both imply the spacetimes have "peeling"–'an idealized property that may not hold in general, as demonstrated in the work of Christodoulou-Klainerman.
- In the following, we describe null infinity and distant observers using the Bondi-Sachs formalism. Our result can be extended to "polyhomogeneity" (Chrusciel-MacCallum-Singleton, 1995) $\mathscr{I}^+$ and the Christodoulou-Klainerman setting.

Bondi-Sachs formalism

- Bondi and his collaborators postulate a coordinate system (Bondi-Sachs) in which the metric tensor is given by

$$-UVdu^2 - 2Ududr + r^2 h_{AB}(dx^A + W^A du)(dx^B + W^B du).$$

- The spacetime is assumed asymptotically flat in the sense that $U, V \to 1$, $h_{AB} \to \sigma_{AB}$, $W^A \to 0$ as $r \to \infty$ and $\det h_{AB} = \det \sigma_{AB}$ (determinant condition) for r large.
- Moreover, outgoing radiation condition is imposed so that all metric coefficients U, V, h_{AB}, W^A can be expanded into power series of $\frac{1}{r}$.
- Bondi et al. found the the physics of gravitational field are encoded in the coefficients of power series expansion.

Bondi-Trautman energy momentum

- Energy and linear momentum are well-understood in the Bondi-Sachs formalism. The expansion

$$V = 1 - \frac{2m(u,x)}{r} + O(r^{-2})$$

gives the *mass aspect* $m(u,x)$ on $\mathscr{I}^+$.

- The Bondi-Trautman energy-momentum $(E, P^k), k = 1, 2, 3$ associated with a $u =$const. slice is

$$E(u) = \int_{S^2} 2m(u,\cdot), \quad P^k(u) = \int_{S^2} 2m(u,\cdot)\tilde{X}^k,$$

where $\tilde{X}^1 = \sin\theta\cos\phi, \tilde{X}^2 = \sin\theta\sin\phi, \tilde{X}^3 = \cos\theta$ are the restriction of standard coordinate functions of $\mathbb{R}^3$ to S^2.

The Bondi mass loss

From the evolution of the mass aspect function, it follows that

$$\frac{d}{du}E(u) = -\frac{1}{4}\int_{S^2} |N|^2 \leq 0$$

The total energy radiated away is then

$$\delta E = E(\infty) - E(-\infty) = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{S^2} |N|^2$$

BMS group and supertranslation

- The symmetry of $\mathscr{I}^+$ is the Bondi-Metzner-Sachs (BMS) group. These are the coordinate transformations at null infinity which keeps the Bondi–Sachs coordinate form.
- There is no preferred coordinate system in a general spacetime. What if we use a different Bondi-Sachs coordinate system?
- For any smooth function $f(x)$ on S^2, the change of coordinates $u = \bar{u} + f(x)$ (on $\mathscr{I}^+$) is called supertranslation.
- It turns out the transformation of the mass aspect and shear tensors under supertranslation are rather complicated. However, the transformation of the news tensor is simple.
- It follows that total energy radiated away computed in u coordinate is the same as in the $\bar{u}$ coordinate. Namely, there is no ambiguity for the energy radiated away.

Large Sphere Limit of Quasilocal Mass at Null Infinity

- Limit of the Wang-Yau quasilocal mass at null infinity recovers the Bondi-Trautman mass.
- For surfaces near the null infinity, the optimal isometric embedding equation has a unique solution.
- For this unique solution, the limit of our quasilocal mass recovers the Bondi-Trautman mass. Non-negativity of the Bondi-Trautman mass follows.
- This process is robust and we carried it out on an asymptotically hyperbolic initial data sets where the null infinity is rougher and more erratic due to radiation. In particular, we don't need any peeling structure of null infinity.

Angular momentum at null infinity

- The definition of angular momentum at null infinity turns out to be more subtle.
- Different approaches (and description of null infinity) lead to different definition by Ashtekar–Hansen, Barnich–Troessaert, Bramson, Chrusciel–Jezierski–Kijowski, Dougan–Manson, Dray–Streubel, Hawking–Perry–Strominger, Ludvigsen–Vickers, Rizzi, Winicour-Tamburino, etc.
- The Dray–Streubel angular momentum of a u cut is defined to be :

$$\tilde{J}(u, Y) = \int_{S^2} Y^A \left(N_A - \frac{1}{4} C_{AB} \nabla_D C^{DB} \right)(u, \cdot)$$

for a rotation Killing field Y. N^A from the expansion of W^A is referred to as the angular momentum aspect.

Supertranslation ambiguity of the angular momentum

- According to Penrose (1982), the very concept of angular momentum gets shifted by these supertranslations and "it is hard to see in these circumstances how one can rigorously discuss such questions as the angular momentum carried away by gravitational radiation".
- In other word, if one choose another Bondi-Sachs coordinates $(\bar{u}, \bar{x})$ related to (u, x) by a supertranslation f, Penrose worries that $\delta \tilde{J}$ may not be the same as $\delta \tilde{J}_f$.
- We shall refer to the difference

$$\delta \tilde{J}_f - \delta \tilde{J}$$

as the supertranslation ambiguity of the angular momentum.

What exactly is the supertranslation ambiguity?

Theorem (Chen–Keller–Wang–Wang–Y,2021)

Suppose the news tensor decays as

$$N_{AB}(u,x) = O(|u|^{-1-\varepsilon})$$

and two Bondi-Sachs coordinate systems are related by a supertranslation f. Then

$$\delta \tilde{J}_f - \delta \tilde{J} = -\int_{S^2} \Big(2fY^A \nabla_A (m(+) - m(-)) \Big)$$

Here

$$m(\pm) = \lim_{u \to \pm\infty} m(u,x)$$

Quasilocal angular momentum

- Transplant a Killing field Y of $\mathbb{R}^{3,1}$ through the unique solution, of the optimal embedding equation, we define the corresponding conserved quantity to be

$$\int_\Sigma -\rho\langle T_0, Y\rangle + j(Y^T)$$

- In particular, the quasilocal angular momentum (Chen-Wang-Y 2015) is

$$\int_\Sigma j(Y^T)$$

- At the quasilocal level, the angular momentum satisfies several important invariant properties, suggesting that their limit at infinity would give total angular momentum with the desired invariant properties.

CWY angular momentum at null infinity

- We computed the limit of the quasilocal angular momentum in Bondi coordinate (Keller-Wang-Y 2018).
- We consider the decomposition of C_{AB} into

$$C_{AB} = \nabla_A \nabla_B c - \frac{1}{2}\sigma_{AB}\Delta c + \frac{1}{2}(\epsilon_A{}^E \nabla_E \nabla_B \underline{c} + \epsilon_B{}^E \nabla_E \nabla_A \underline{c})$$

- The CWY angular momentum associated to a rotation Killing field is

$$J(u, Y) = \int_{S^2} Y^A \left(N_A - \frac{1}{4} C_{AB} \nabla_D C^{DB} - c \nabla_A m \right)(u, \cdot)$$

- One should note that c has never occurred in any previous definition of angular momentum. It is a "non-local term"

Supertranslation invariance of the new angular momentum

Theorem (Chen-Keller-Wang-Wang-Y, 2021)

Suppose $N_{AB}(u, \cdot) = O(|u|^{-1-\epsilon})$ as $|u| \to \infty$. Under a supertranslation $f = \alpha_0 + \alpha_i \tilde{X}^i + f_{l \geq 2}$, the total flux of the CWY angular momentum

$$\delta J(Y) = J(+\infty, Y) - J(-\infty, Y)$$

transforms according to

$$(\delta J)_f - \delta J = \alpha_i \epsilon^{ik}{}_j \delta P^j \text{ for } Y^A = \epsilon^{AB} \nabla_B \tilde{X}^k,$$

where δP^j is the total flux of Bondi-Sachs linear momentum.

In particular, δJ is invariant under pure supertranslation and the transformation law is the same as the special-relativistic angular momentum.

Cross-Section Continuity of Definitions of Angular Momentum

- Finally, we discuss the "cross-section continuity" criterion of angular momentum. Namely, if a sequence of cross-sections of null infinity converges continuously to a given cross-section, does the angular momentum converges too.
- Such a property is not immediate obvious since the definition of angular momentum depends on higher order derivatives.
- Nevertheless, a definition of the angular momentum is not physically viable if such criterion does not hold.
- The Dray-Streubel (DS) definition of angular momentum satisfies this criterion by the flux formula of Wald-Zoupas.
- While the CWY angular momentum contains non-local terms, it also satisfies this criterion (Chen-Paraizo-Wald-Wang-Wang-Y.)

Concluding remarks

We obtain a complete set of ten conserved quantities (E, P^k, J^k, C^k) at null infinity (all as functions of the retarded time u) that satisfy the following properties:

- (E, P^k, J^k, C^k) all vanish for any Bondi-Sachs coordinate system of the Minkowski spacetime where there is no gravity.
- In a Bondi-Sachs coordinate system of the Kerr spacetime, P^k and C^k vanish, and E and J^k recover the mass and angular momentum.
- On a general spacetime, the total fluxes of (E, P^k, J^k, C^k) are supertranslation invariant.
- (E, P^k, J^k, C^k) and their fluxes transform according to basic physical laws under ordinary translations.