Weak saturation number of the 3-cube in the complete graph

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Weak saturation number

Weak saturation graph

A weak saturation graph $\mathrm{WSAT}(n, H)$ is a graph K with n vertices and minimum number of edges such that there is a sequence of edges $e_1, \ldots, e_k$ such that for each $i = 0, \ldots, k-1$, there is a copy of H in $K \cup \{e_1, \ldots, e_{i+1}\}$ which is not in $K \cup \{e_1, \ldots, e_i\}$ and $K \cup \{e_1, \ldots, e_k\} = K_n$.

Weak saturation number

The weak saturation number $\mathrm{wsat}(n, H)$ is the number of edges of $\mathrm{WSAT}(n, H)$.

Approach: Subgraph-supergraph

Lemma

If $H \subseteq G$, then $\mathrm{wsat}(n, H) \leq \mathrm{wsat}(n, G)$.

Subgraphs of Q_3

- $\mathrm{wsat}(n, C_4) = n$ for $n \geq 4$.
- $\mathrm{wsat}(n, 2\text{ faces of cubes}) = n + 2$ for $n \geq 6$.
- $\mathrm{wsat}(n, 3\text{ faces of cubes}) = n + 2$ for $n \geq 7$.

In general, this approach can only give a lower bound of constant coefficient 1.

Approach: Subgraph-supergraph

Supergraph of Q_3

$\mathrm{wsat}(n, K_{4,4}) = 3(n-1)$ for $n \geq 9$ (Kronenberg-Martins-Morrison '21).

This approach allows us to narrow the bounds to

$$n + 2 \leq \mathrm{wsat}(n, Q_3) \leq 3(n-1)$$

for $n \geq 9$.

Approach: Computation

Using careful computation in Python, we obtained

$$\mathrm{wsat}(8, Q_3) = 15$$

and

$$\mathrm{wsat}(9, Q_3) = 17.$$

Our approach: to test if $\mathrm{wsat}(n, Q_3) \le m$, we can fix a "canonical" cube which is fully filled after the first weak saturation edge, and consider all $\binom{\binom{n}{2}-12}{m-12}$ possible configurations of other initial edges. For each configuration, we use dynamic programming on the subset of edges outside the canonical cube to find if we can reach K_n, which has $O(\binom{n}{2}2^{\binom{n}{2}-m})$ time complexity per configuration.

Approach: Constructing with minimal degree vertex

Lemma

Let H be a graph and $\delta(H)$ denote the minimal degree of H.
Then $\mathrm{wsat}(n+1, H) \leq \mathrm{wsat}(n, H) + \delta(H) - 1$ for all $n \geq |V(H)|$.

Combined with the previous computation, this yields

$$\mathrm{wsat}(n, Q_3) \leq 2n - 1$$

for all $n \geq 8$.

Approach: Linear algebra

Lemma (Balogh-Bollobás-Morris-Riordan '12)

Let H be graph and W be vector space. Suppose that there is a set $\{f_e : e \in E(K_n)\} \subseteq W$ such that for every copy H' of H in K_n and edge $e \in H'$, there is a scalar $c_{H',e} \neq 0$ for which $\sum_{e \in H'} c_{H',e} f_e = 0$. Then

$$\mathrm{wsat}(n, H) \geq \dim\langle f_e : e \in E(K_n)\rangle$$

Improved lower bound

Theorem (Terekhov-Zhukovskii '25)

If $\delta(H)$ is even and H is 2-edge-connected, then, for $n \geq |V(H)|$,

$$\mathrm{wsat}(n, H) \geq \frac{\delta(H)}{2}(n - |V(H)|) + e(H) - 1.$$

In our case, we have a lower bound of

$$\mathrm{wsat}(n, H) \geq \frac{3}{2}n - 1.$$

Approach: Prisms

A more general quantity we hope to compute is $\mathrm{wsat}(n, Q_m)$ for all m.

Definition

Given a graph G, let $\mathrm{Prism}(G)$ denote the graph which consists of two copies of G with disjoint vertex sets and an edge between each vertex and its copy.

If we can relate $\mathrm{wsat}(n, \mathrm{Prism}(G))$ to $\mathrm{wsat}(n, G)$, then we can extrapolate $\mathrm{wsat}(n, Q_m)$ for all m from the known

$$\mathrm{wsat}(n, C_4) = n.$$

Prisms continued: Graph duplication

The prism is difficult, so we also study disjoint duplicates.

Definition

Given a graph G, let $\mathrm{Dup}(G)$ denote the graph consisting of two disjoint copies of G.

Proposition

For all graphs G and all sufficiently large n,

$$\mathrm{wsat}(n, G) \leq \mathrm{wsat}(n, \mathrm{Dup}(G)) \leq \mathrm{wsat}(n, G) + |E(G)|.$$

Proposition

For $n \geq 12$, $\mathrm{wsat}(n, \mathrm{Dup}(K_3)) = \mathrm{wsat}(n, K_3) + 1$.

Results

Proposition

For all $n \geq 8$, $\frac{3}{2}n - 1 \leq \text{wsat}(n, Q_3) \leq 2n - 1$.

The upper bound is sharp for $n = 8$ and $n = 9$. Among other reasons, this leads us to believe

Conjecture

For all $n \geq 8$, $\text{wsat}(n, Q_3) = 2n - 1$.