Problem Set 5: The Power of Classical

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Problem 1 ([CGLLTW22, Lemma 4.9])**.** Given $\mathrm{SQ}(A) \in \mathbb{C}^{m\times n}$ and $\varepsilon \in (0,1]$, we can form importance sampling sketches $S \in \mathbb{R}^{r\times m}$ and $T^\dagger \in \mathbb{R}^{c\times n}$ in $\mathcal{O}(rc\,\mathbf{sq}(A))$ time. Let σ_i and $\hat{\sigma}_i$ denote the singular values of A and SAT, respectively (where $\hat{\sigma}_i = 0$ for $i > \min(r,c)$). How big does our sketch ($r \times c$) need to be for the following property to hold with probability 0.9?

$$\Big(\sum_{i=1}^{\min(m,n)} (\hat{\sigma}_i^2 - \sigma_i^2)^2\Big)^{1/2} \leq \varepsilon \|A\|_{\mathrm{F}}^2. \tag{$\star$}$$

Problem 2 ([CGLLTW22, Corollary 6.12])**.** We now show that the previous problem implies a dequantization of QPCA [LMR14]. Given a matrix $\mathrm{SQ}(X) \in \mathbb{C}^{m\times n}$ such that $X^\dagger X$ has top k eigenvalues $\{\lambda_i\}_{i=1}^k$, along with a lower bound ν such that $\lambda_1,\ldots,\lambda_k \geq \nu$, compute eigenvalue estimates $\{\hat{\lambda}_i\}_{i=1}^k$ such that, with probability 0.9,

$$\sum_{i=1}^k |\hat{\lambda}_i - \lambda_i| \leq \varepsilon\,\mathrm{tr}(X^\dagger X). \tag{1}$$

What is the runtime of this classical algorithm?

Bonus: how would you design a quantum algorithm to solve this task? Suppose we are given a state prep unitary that prepares a purification of $\rho = X^\dagger X$ (i.e. the vectorized version of X), which implies both the ability to prepare ρ and a 1-block encoding of ρ.

Problem 3 ([Van11; GL22])**.** Suppose we are given a classical description of an n-qubit product state $|\psi\rangle$ and a description of $H = \frac{1}{k}\sum_{i=1}^k \lambda_a E_a$, where $\lambda_a \in [-1,1]$ and E_a are Pauli matrices. Show how to estimate $\langle\psi|\, H^k\, |\psi\rangle$ to ε error in $\mathrm{poly}(n, s^k, 1/\varepsilon)$ time.

Bonus: prove you can still perform the above estimate if $|\psi\rangle$ is a matrix product state with polynomial bond dimension, meaning that, for some $2n$ $\mathrm{poly}(n) \times \mathrm{poly}(n)$ matrices $A_i[0], A_i[1]$, $\psi_{b_1\cdots b_n} = \mathrm{tr}(A_1[b_1]\cdots A_n[b_n])$. Here, $b_1\cdots b_n$ are bits.

References

[CGLLTW22] Nai-Hui Chia, András Pal Gilyén, Tongyang Li, Han-Hsuan Lin, Ewin Tang, and Chunhao Wang. “Sampling-based sublinear low-rank matrix arithmetic framework for dequantizing quantum machine learning”. In: *Journal of the ACM* 69.5 (Oct. 2022), pp. 1–72. DOI: 10.1145/3549524. arXiv: 1910.06151 [cs.DS] (page 1).