Problem Set 2: Proving the QSVT

July 25, 2023

Problem 1 (When will my reflection show who I am inside?)**.** QSVT achieves polynomials by interspersing phase operators with signal rotation operators. However, these rotation operators may look different in the literature. Consider two potential operators, $W(x), R(x)$, with the following matrix forms:

$$W(x) = \begin{bmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{bmatrix} \qquad R(x) = \begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{bmatrix} \tag{1}$$

Where W is the rotation operator while R is the reflection operator. We can define two different kinds of QSP, $\mathbf{QSP}_W(\Phi, x)$ and $\mathbf{QSP}_R(\Phi, x)$ for these two different operators. For example,

$$\mathbf{QSP}_W(\Phi, x) \coloneqq \Big(\prod_{j=1}^{n} e^{\mathrm{i}\phi_j \sigma_z} W(x)\Big) e^{\mathrm{i}\phi_0 \sigma_z}.$$

Suppose we have some series of phases $\Phi = (\phi_0, \dots, \phi_n)$ such that $\mathbf{QSP}_W(\Phi, x)$ forms a desired polynomial $p(x)$. Can we find a Φ' such that $\mathbf{QSP}_R(\Phi', x)$ performs the same polynomial? If so, find a formula for Φ' in terms of Φ; if not, prove why.

Problem 2 (Perfectly balanced, as all things should be)**.** The Chebyshev polynomials of the first and second kind are functions such that, for all $z \in \mathbb{C}$,

$$T_n(\tfrac{1}{2}(z + z^{-1})) = \tfrac{1}{2}(z^n + z^{-n})$$
$$U_n(\tfrac{1}{2}(z + z^{-1})) = (z^{n+1} - z^{-(n+1)})/(z - z^{-1})$$

Prove that T_n and U_n are polynomials. Then, prove that

$$T_n(x)^2 + (1 - x^2)U_n(x)^2 = 1. \tag{2}$$

Just a little more and we have a proof that these can be used in QSP/QSVT!

Problem 3 (They're the same picture!)**.** Return to [BCCKS17, Lemma 3.6]. What are the angles of the phase operators? What are the polynomials that are being computed with these phase operators? (A recursive definition is fine.)

Problem 4 (Block-encodings for any matrix)**.** Given a matrix $A \in \mathbb{C}^{d\times d}$ such that $\|A\| \leq 1$, show there exists a unitary $U \in \mathbb{C}^{2d\times 2d}$ such that U is a block-encoding of A:

$$U = \begin{pmatrix} A & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

Prove that $2d$ is tight, i.e., there is some matrix A such that any unitary with A as a submatrix must be size at least $2d \times 2d$. *Note: this is true for non-square A as well, but the argument might get more annoying.*

Problem 5 (It's just a phase)**.** In our QSVT algorithm, we needed to apply gates of the form $e^{\mathrm{i}\phi(2\Pi - I)}$, where $\Pi = (|0\rangle^{\otimes a} \langle 0|^{\otimes a}) \otimes I$. How do you implement these?