Problem Set 3: Polynomial Approximation

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Problem 1 (Polynomial approximation of monomials)**.** First, compute the Chebyshev coefficients of the monomial $m^{(n)}(x) = x^n$. (Doing this via $T_k(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^n + z^{-n})$ formulation may be easiest.) How small can k be such that the Chebyshev truncation $m_k^{(n)}$ a good approximation of $m^{(n)}$:

$$\|m^{(n)} - m_k^{(n)}\|_{[-1,1]} \leq \varepsilon?$$

Problem 2 (Chebyshev interpolation [Tre19])**.** The *Chebyshev interpolant* of a function f, denoted p_d, is the unique degree-d polynomial such that $p_d(x_j) = f(x_j)$ for all $x_j = \cos(j\pi/d)$, $j = 0, 1, \ldots, d$. Prove that[1]

$$\|f(x) - p_d(x)\|_{[-1,1]} \leq 2 \sum_{\ell \geq d} |a_d|.$$

Hint: when is $T_k(x_j) = T_\ell(x_j)$ for all points $\{x_j\}$?

Problem 3 (Jackson theorems, [Tre19])**.** Let $f : [-1, 1] \to \mathbb{R}$ be absolutely continuous and suppose f is of bounded variation, meaning that $\int_{-1}^{1} |f'(x)| \mathrm{d}x \leq V$. Then show that the Chebyshev coefficients of f satisfy

$$|a_k| \leq \frac{2V}{\pi k}.$$

Problem 4 (Optimal polynomial approximations; upper and lower bounds)**.** Consider a function $f : [-1, 1] \to \mathbb{R}$ with a Chebyshev expansion $f(x) = \sum_{k \geq 0} a_k T_k(x)$. Prove that

$$\Big(\frac{1}{2} \sum_{k=n+1}^{\infty} a_k^2\Big)^{\frac{1}{2}} \leq \min_{\substack{p \in \mathbb{R}[x] \\ \deg p = n}} \|f(x) - p(x)\|_{[-1,1]} \leq \sum_{k=n+1}^{\infty} |a_k|$$

For what kind of Chebyshev coefficient decay is this characterization tight up to constants?

References

[Tre19] Lloyd N. Trefethen. *Approximation theory and approximation practice, extended edition*. Extended edition [of 3012510]. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2019, pp. xi+363. ISBN: 978-1-611975-93-2. DOI: 10.1137/1.9781611975949 (page 1).

[1] Recall that our approximation results used that $\|f(x) - f_d(x)\|_{[-1,1]} \leq \sum_{\ell \geq d} |a_d|$. So, Chebyshev interpolants p_d give the same results as Chebyshev truncations f_d, up to a constant factor. Interpolants have the advantage of being computable in $d + 1$ function evaluations.