Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Statistics of Vortex Surfaces I. Stability and Stochasticity.

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Stable Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Abstract

We offer a new approach to the theory of vortex sheet turbulence. The origin of irreversibility is the microscopic stability of the vortex sheet, leading to new boundary conditions for the local strain at the surface, restricting its shape. The position and scale of the vortex surface remains arbitrary. The scale is related to the energy dissipation, but the position remains as a zero mode. These random positions in ensemble of closed vortex surfaces distributed in infinite space, lead to spontaneous stochasticity. The Gaussian random background strain tensor is created self-consistently as a mean field in statistical mechanics and it determines statistics of vortex structures. We prove the conservation of the surface energy dissipation in the Navier-Stokes equation under these boundary conditions and present exact solutions.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The following general ansatz [1] describes the vortex sheet vorticity:

$$\vec{\omega}(\vec{r}) = \int_{\Sigma} d\vec{\Omega}\delta^3\left(\vec{X} - \vec{r}\right) \tag{1}$$

where the 2-form

$$d\vec{\Omega} \equiv d\Gamma \wedge d\vec{X} = d\xi_1 d\xi_2 e_{ab} \frac{\partial \Gamma}{\partial \xi_a} \frac{\partial \vec{X}}{\partial \xi_b}; \tag{2}$$

This vorticity is zero everywhere in space, except the surface, where it is infinite. To describe the physical vorticity of the fluid, this ansatz must satisfy the divergence equation (the conservation of the "current" $\vec{\omega}$ in the language of statistical field theory)

$$\vec{\nabla} \cdot \vec{\omega} = 0; \tag{3}$$

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

This relation is built into this ansatz for arbitrary $\Gamma(\xi)$, as can be verified by direct calculation. In virtue of the singular behavior of the Dirac delta function, it may be easier to understand this calculation in Fourier space

$$\vec{\omega}^F(\vec{k}) = \int d^3r e^{\imath\,\vec{k}\cdot\vec{r}}\vec{\omega}(\vec{r}) = \int_\Sigma d\vec{\Omega} e^{\imath\,\vec{k}\cdot\vec{X}}; \tag{4}$$

$$\imath\,\vec{k}\cdot\vec{\omega}^F(\vec{k}) = \int_\Sigma d\Gamma \wedge d\vec{X}\cdot(\imath\,\vec{k})e^{\imath\,\vec{k}\cdot\vec{X}} =$$

$$\int_\Sigma d\Gamma \wedge de^{\imath\,\vec{k}\cdot\vec{X}} = \int_{\partial\Sigma} d\Gamma e^{\imath\,\vec{k}\cdot\vec{X}}; \tag{5}$$

In case there is a boundary of the surface, this $\Gamma(\xi)$ must be a constant at the boundary for the identity $\vec{k}\cdot\omega^F(\vec{k}) = 0$ to hold.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

It may be instructive to write down an explicit formula for the tangent components of vorticity in the local frame, where x, y is a local tangent plane and z is a normal direction

$$\omega_j(x, y, z) = \partial_i \Gamma e_{ij} \delta(z); \tag{6}$$

$$\omega_z(x, y, z) = 0; \tag{7}$$

In particular, outside the surface, $\vec{\omega} = 0$, so that its divergence vanishes trivially.

The divergence is manifestly zero in this coordinate frame

$$\vec{\nabla} \cdot \vec{\omega} = \delta(z) \partial_j \partial_i \Gamma e_{ij} = 0; \tag{8}$$

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Let us compare this with the Clebsch representation

$$\omega_\alpha = e_{\alpha\beta\gamma}\partial_\beta\phi_1\partial_\gamma\phi_2, \tag{9}$$

We see that that in case ϕ_2 takes one space-independent value ϕ_2^{in} inside the surface and another space-independent value ϕ_2^{out} outside, the vorticity will have the same form, with

$$\Gamma = \phi_1(\phi_2^{in} - \phi_2^{out}) \tag{10}$$

Thus, the closed vortex sheets are bubbles of confined constant Clebsch field.

The vortex surfaces with a boundary curve [2] are more complex topological objects made out of the Clebsch field on a unit sphere S_2.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The surface is driven by the self-generated velocity field (arithmetic mean of velocity above and below the surface).

Let us substitute our ansatz for vorticity into the Biot-Savart integral for the velocity field and change the order of integration

$$\vec{v}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla} \times \int d^3r' \frac{1}{|\vec{r} - \vec{r'}|} \int d\vec{\Omega}\delta^3(X - \vec{r'}) = \frac{1}{4\pi} \int d\vec{\Omega} \times \vec{\nabla} \frac{1}{|\vec{r} - \vec{X}|} \tag{11}$$

The tangent gap in velocity is directly related to the gradient of the potential gap $\Gamma = \Phi_+ - \Phi_-$

$$\Delta\vec{v} = \vec{\nabla}\Gamma \tag{12}$$

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Lagrange equations of motion for the surface

$$\partial_t \vec{X}(\xi) = \vec{v}\left(\vec{X}(\xi)\right); \tag{13}$$

$$\partial_t \Gamma = 0 \tag{14}$$

were shown in [1], [3] to follow from the action

$$S = \int \Gamma dV - \int H dt; \tag{15}$$

$$dV = d\xi_1 d\xi_2 dt \frac{\partial \vec{X}}{\partial \xi_1} \times \frac{\partial \vec{X}}{\partial \xi_2} \cdot \partial_t \vec{X}; \tag{16}$$

$$H = \frac{1}{2}\int d^3 r \vec{v}^2 = \frac{1}{2}\int_S \int_S \frac{d\vec{\Omega} \cdot d\vec{\Omega}'}{4\pi |\vec{X} - \vec{X}'|}; \tag{17}$$

This dV is the 3-volume swept by the surface area element in its movement for the time dt.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The easiest way to derive the vortex sheet representation for the Hamiltonian is to go in Fourier space where the convolution becomes just a multiplication and use the incompressibility condition $\vec{k} \cdot \vec{v}^F(\vec{k}) = 0$

$$\vec{\omega}^F(\vec{k}) = \imath\, \vec{k} \times \vec{v}^F(\vec{k}); \tag{18}$$

$$\vec{v}^F(\vec{k}) \cdot \vec{v}^F(-\vec{k}) = \frac{\vec{\omega}^F(\vec{k}) \cdot \vec{\omega}^F(-\vec{k})}{\vec{k}^2} \tag{19}$$

In the case of the handle on a surface, Γ acquires extra term $\Delta\Gamma = \oint_\gamma \Delta\vec{v} \cdot d\vec{r}$ when the point goes around one of the cycles $\gamma = \{\alpha, \beta\}$ of the handle.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces

Hamiltonian Dynamics of Vortex Surfaces

Attractor

Zero Modes

Irreversibility

Enstrophy

Stationary Vortex Sheets and Double-layer potentials

CVS as a variational problem

The induced background strain

Energy dissipation and its distribution

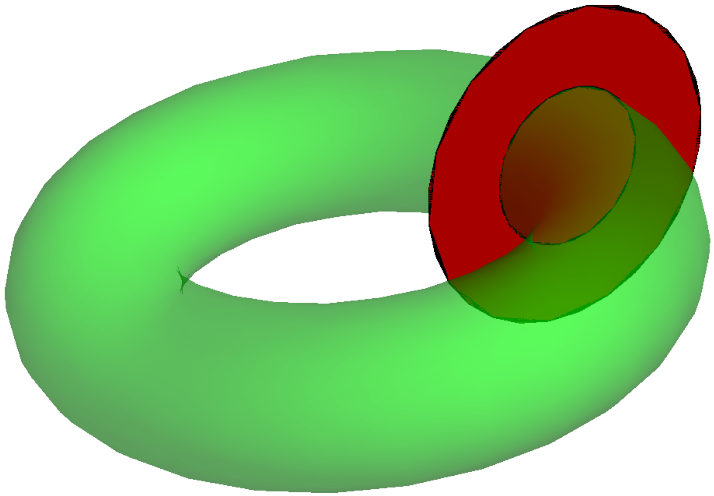


Figure: The green vortex bubble T cut vertically by a red disk D_C. The vorticity flux through the disk reduces to integral of velocity discontinuity over α cycle.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

This $\Delta\Gamma$ does not depend on the path shape because there is no normal vorticity at the surface, and thus there is no flux through the surface. This topologically invariant $\Delta\Gamma$ represents the flux through the handle cross-section.

This ambiguity in Γ makes our action multivalued as well.

Let us check the equations of motion emerging from the variation of the surface at fixed Γ:

$$\delta \int H dt = \int d\vec{\Omega} \times \delta\vec{X} \cdot \vec{v}(\vec{X}) dt; \tag{20}$$

$$\delta \int \Gamma dV = \int d\vec{\Omega} \times \delta\vec{X} \cdot \partial_t \vec{X} dt \tag{21}$$

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

As we already discussed above, the tangent components of velocity at the surface create tangent motion, resulting in the surface's re-parametrization.

One of the two tangent components of the velocity (along the line of constant $\Gamma(\xi)$) does not contribute to variation of the action, so that the correct Lagrange equation of motion following from our action reads

$$\partial_t \vec{X}(\xi) = \vec{v}(\vec{X}(\xi)) \mod e^{ij}\partial_i \Gamma \partial_j \vec{X} \tag{22}$$

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

We noticed this gauge invariance before in[3]; now we see that as both tangent components of the velocity only lead to the re-parametrization of a surface, they do not represent an observable change of the surface.

However, the normal component of the velocity must vanish in a steady solution, which provides a linear integral equation for the conserved function $\Gamma(\xi)$. This equation corresponds to **global** minimization of the Hamiltonian by Γ in the stable, steady solution of the Euler equations.

As it was noticed this year (see below), there are additional stability conditions (*CVS*) restricting the shape of the surface as well as the velocity field around it.

Hamiltonian Dynamics of Vortex Surfaces

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The flow at the scales larger than the width of the boundary layer (see below) obeys the Euler equations and is purely potential.

The vorticity is concentrated in the boundary layer, and at these scales, one has to solve full Navier-Stokes equations.

However, at these scales, any smooth surface can be treated as a local tangent plane, with the local value of the strain as a background for the solution of the Navier-Stokes equation with planar vortex sheet.

The stability of this solution provides the boundary condition involving the local strain tensor and the normal vector to the surface.

Attractor

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Specifically, the local potential velocity $\vec{v}_\pm = \vec{\nabla}\Phi_\pm$ and local strain tensor $\hat{S}_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ must satisfy the *CVS* boundary conditions

$$\vec{n}_0 \cdot \vec{v}_\pm(\vec{r}_0) = 0; \ \forall \vec{r}_0 \in \mathcal{T}; \tag{23}$$

$$\hat{S}(\vec{r}_0) \cdot \Delta\vec{v}(\vec{r}_0) = 0; \tag{24}$$

$$\vec{n}_0 \cdot \hat{S}(\vec{r}_0) \cdot \vec{n}_0 < 0 \tag{25}$$

These conditions as well as the shape of the Burgers-Townsend solution [4], [5] follow from the analysis of the Navier-Stokes equations [6], [7].

They are not satisfied with an arbitrary surface but rather define the family of stable vortex surfaces.

Attractor

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

We imply the boundary condition of some constant background strain $\hat{S}(\infty)$ at spacial infinity. It should be distinguished from the local strain $\hat{S}(\vec{r}_0)$ at every point r_0 at the surface.

The CVS equation demands that this local strain be degenerate (middle eigenvalue=0) at every point of the surface for arbitrary background strain.

So, in case the background strain is degenerate, as in Burgers-Townsend solution, the surface would be a plane, as there is no need to curve it for stability.

However, if the strain at infinity is nondegenerate, the surface rotates and bends. This distortion of the surface changes the local velocity field to kill the projection of the strain on the velocity gap. Otherwise, the sheet would leak vorticity.

Attractor

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Navier-Stokes solution in the local tangent plane looks as follows. The velocity gap $\Delta\vec{v}$ is directed along x, vorticity along y and normal to the plane being z in coordinate frame where strain is diagonal: $\hat{S} = \{S_x, S_y, S_z\}$. Only the degenerate case $S_x = 0$ is stable.
For the generic strain the Navier-Stokes equation is solved by the following vorticity

$$\omega_y \propto \Delta v \exp\left(-\eta^2\right) H_{\frac{S_x}{S_z}}(\eta); \tag{26}$$

$$\eta = z\sqrt{\frac{-S_z}{2\nu}}; \tag{27}$$

$$\tag{28}$$

where $H_\mu(\eta)$ is a Hermite polynomial $H_n(z)$, analytically continued to arbitrary index μ as a combination of hypergeometric functions [6], [7].

Attractor

Stable Vortex Surfaces

Hamiltonian Dynamics of Vortex Surfaces

Attractor

Zero Modes

Irreversibility

Enstrophy

Stationary Vortex Sheets and Double-layer potentials

CVS as a variational problem

The induced background strain

Energy dissipation and its distribution

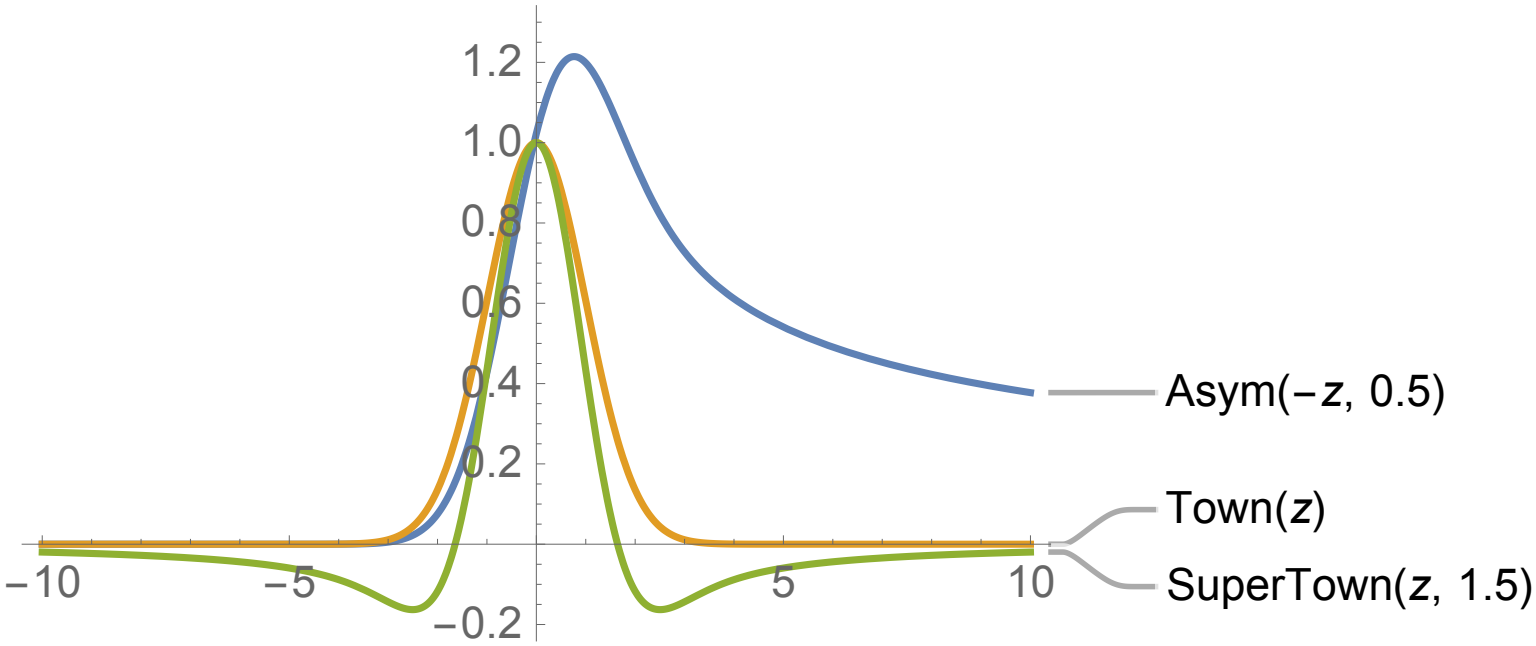


Figure: The vorticity profiles for asymmetric, Townsend and super-Townsend strains

Zero Modes

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Euler equation as well as the *CVS* boundary condition have no spatial scale and they are invariant with respect to translations and rotations.

The boundary condition $\hat{S}(\infty)$ provides the time scale and breaks rotational invariance. However, the spacial scale R_0 as well as the origin position $\vec{r}_c$ remain as zero modes.

The general solution of the *CVS* equations in presence of the background strain tensor $\hat{S}(\infty)$ has the form

$$S : \vec{r} = \vec{r}_0 + R_0 \vec{R}(u, v); \tag{29}$$

$$\vec{v} = \vec{v}_0 + \lambda_{max}(\hat{S}(\infty)) R_0 \vec{F}\left(\frac{\vec{r} - \vec{r}_0}{R_0}\right); \tag{30}$$

where $\lambda_{max}(\hat{S}(\infty))$ is the largest eigenvalue of $\hat{S}(\infty)$.

Irreversibility

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The velocity gap is an approximation to the stable solution [8], which is valid inside the boundary layer of the bubble.

$$\vec{v}\left(\vec{r}_0 + \vec{n}_0\zeta\right) = \vec{v}_R(\vec{r}_0) + \frac{1}{2}\Delta\vec{v}(\vec{r}_0)\,\mathrm{erf}\left(\frac{\zeta}{h\sqrt{2}}\right); \quad (31)$$

$$\vec{n}_0 = \vec{n}(\vec{r}_0); \quad (32)$$

$$\vec{v}_R(r) = \frac{1}{2}\vec{\nabla}\Phi_+(\vec{r}_0) + \frac{1}{2}\vec{\nabla}\Phi_-(\vec{r}_0); \quad (33)$$

$$\vec{\omega}\left(\vec{r}_0 + \vec{n}_0\zeta\right) = \frac{\sqrt{2}}{h\sqrt{\pi}}\vec{n} \times \Delta\vec{v}(\vec{r}_0)\exp\left(-\frac{\zeta^2}{2h^2}\right); \quad (34)$$

$$h = \sqrt{\frac{\nu}{-\vec{n}_0 \cdot \hat{S}(\vec{r}_0) \cdot \vec{n}_0}}; \quad (35)$$

$$\hat{S}_{\alpha\beta}(\vec{r}_0) = \frac{1}{2}\partial_\alpha\partial_\beta\Phi_+(\vec{r}_0) + \frac{1}{2}\partial_\alpha\partial_\beta\Phi_-(\vec{r}_0) \quad (36)$$

Irreversibility

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

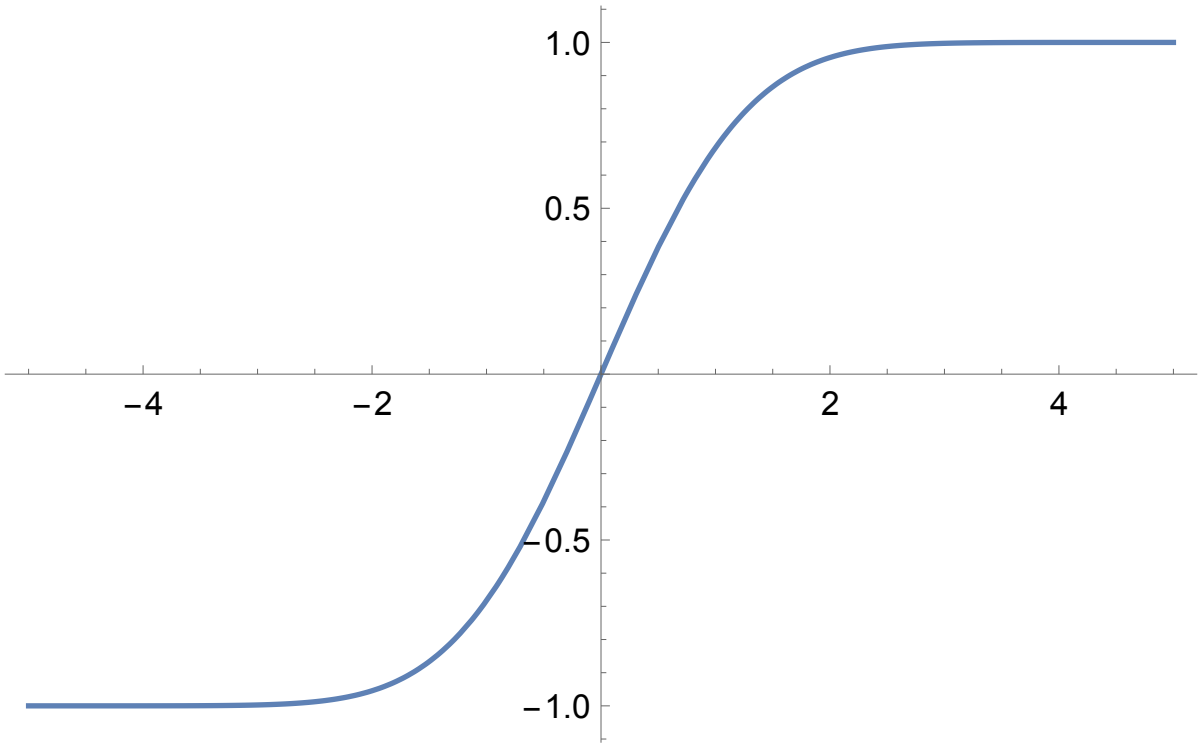


Figure: The velocity gap profile in the boundary layer (the Error function).

Irreversibility

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Euler equation is invariant under time reversal, changing the sign of the strain. Without the *CVS* boundary conditions, both signs of the normal strain would satisfy the stationary Euler equation. Therefore, this *CVS* vortex sheet represents a dynamical breaking of the time-reversal symmetry.

Out of the two time-reflected solutions of the Euler equation, only the one with the negative normal strain survives. If virtually created as a metastable phase, the other one dissolves in the turbulent flow, but this remains stable.

Technically this instability displays itself in the lack of the real solutions of the steady Navier-Stokes equation for positive S_{nn}.

Irreversibility

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Gaussian profile of vorticity as a function of normal coordinate formally becomes complex at positive S_{nn}, which means instability or decay in the time-dependent equation.

In [7] the authors verified this decay/instability process. The time-dependent Navier-Stokes equation was solved numerically in the vicinity of the steady solution with arbitrary background strain. Only the Burgers-Townsend solution corresponding to our *CVS* conditions on the strain was stable.

Irreversibility

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The background strain $\hat{S}(\infty)$ does not define the sign of the solution of the *CVS* equation, though in the Laplace equation for the potential the boundary strain enters linearly.

The constant traceless matrix $\hat{S}(\infty)$ has three ordered eigenvalues $a \leq b \leq c$,such that $a + b + c = 0$.

Obviously $a \leq 0, c \geq 0$ with equality only possible when all eigenvalues are zero. The time reflection $\hat{S}(\infty) \Rightarrow -\hat{S}(\infty)$ corresponds to $c \Rightarrow -a$.

The third *CVS* equation $S_{nn} < 0$ selects only one of two solutions with respect to time reversal. Thus, the time reversibility is spontaneously broken by the Navier-Stokes stability condition.

Enstrophy

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

There is another manifestation of the stability of the *CVS* vortex surface.

We have also shown [9] that given these boundary conditions, the energy dissipation integral

$$\mathcal{E} = \nu \int d^3 r \vec{\omega}^2 \tag{37}$$

is confined to the vortex sheet $\mathcal{S}$

$$\mathcal{E} \to \frac{\sqrt{\nu}}{2\sqrt{\pi}} \int_{\mathcal{S}} dS \sqrt{-\vec{n} \cdot \hat{S} \cdot \vec{n}} \; (\Delta \vec{v})^2 \, ; \tag{38}$$

Enstrophy

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The time derivative of enstrophy vanishes for any steady solution. It is instructive to check how it happens for the Burgers-Townsend solution on the CVS surface. Integrating out the normal coordinate locally we find for time derivative on the steady solution

$$\partial_t \mathcal{E} =$$

$$\nu \int dS \int_{\infty}^{\infty} dz \left(-v_\alpha \partial_\alpha \omega_\alpha^2 + 2\omega_\alpha \omega_\beta \partial_\beta v_\alpha + 2\nu \omega_\alpha \partial_\beta^2 \omega_\alpha\right) \rightarrow$$

$$\frac{\sqrt{\nu}}{2\sqrt{\pi}} \int dS \sqrt{S_k^k} \Delta\tilde{v}^i \Delta\tilde{v}^j \left(S_{ij} - S_k^k g_{ij}\right); \tag{39}$$

$$\tilde{v}^i = e^{ij} v_j;\ dS = d^2\xi\sqrt{G};\ g_{ij} = \partial_i \vec{R}(\xi) \cdot \partial_j \vec{R}(\xi) \tag{40}$$

Enstrophy

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The advection, vortex stretching and viscose diffusion terms combine into a single formula

$$(\partial_t \mathcal{E})_{NS} = -\frac{\sqrt{\nu}}{2\sqrt{\pi}} \int dS \sqrt{-\vec{n}\cdot\hat{S}\cdot\vec{n}}\, \Delta\vec{v}\cdot\hat{S}\cdot\Delta\vec{v} = 0 \qquad (41)$$

which vanishes at every point of the surface in virtue of the *CVS* conditions.

For a generic vortex surface the velocity linearly grows and vorticity decreases as a power outside the surface. Therefore there would be a finite enstrophy flux through any surface S' surrounding the vortex surface

$$\int_{V':\partial V'=S'} d^3 r v_\alpha \partial_\alpha \omega_\beta^2 = \int_{S'>S} d\vec{\sigma}\cdot\vec{v}\vec{\omega}^2 \neq 0 \qquad (42)$$

For *CVS* this flux is exponentially small outside the boundary layer so that both the enstrophy and its time derivative are concentrated on the surface.

Stationary Vortex Sheets and Double-layer potentials

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Let us go into details of the *CVS* equations.
In the 3D space inside and outside the surface $\mathcal{S}^{\pm}$: $\partial\mathcal{S}^{\pm} = \mathcal{S}$ there is no vorticity so that the flow can be described by a potential $\Phi_{\pm}(\vec{r})$ with the gap on the surface.
This is a well-known double-layer potential from electrostatics [10].

$$\Phi_{\pm}(\vec{r}) = \frac{1}{2} W_{\pm}^{\alpha\beta} r_\alpha r_\beta - \frac{1}{4\pi} \int_S \Gamma_{\pm}(\vec{r}') e_{\alpha\beta\gamma} dr_\beta \wedge dr_\gamma \partial_\alpha \frac{1}{|\vec{r} - \vec{r}'|}; \tag{43}$$

The pressure in each of the domains inside/outside is given by the Bernoulli formula

$$p_{\pm} = -\frac{1}{2} \left(\partial_\alpha \Phi_{\pm}(\vec{r})\right)^2 \tag{44}$$

Stationary Vortex Sheets and Double-layer potentials

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The normal velocity vanishes on both sides (Neumann boundary conditions for $\Phi_{\pm}$).

The tangent velocity, on the other hand, has a gap $\Delta\vec{v}$. This gap arises because of the gap in the potential

$$\Delta\vec{v} = \vec{\nabla}\Phi_{+}(\vec{r}) - \vec{\nabla}\Phi_{-}(\vec{r}) \tag{45}$$

In that case, there are 5 unknown functions and 5 *CVS* equations (two normal derivatives and one vector equation for the projection of the strain on the velocity gap).

CVS as a variational problem

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The steady solution of the vortex sheet dynamics minimizes the Hamiltonian

$$H_{tot} = H[\Gamma_+, \mathcal{S}; \hat{W}_+] + H[\Gamma_-, \mathcal{S}; \hat{W}_-] + \int_{\vec{r}\in\mathcal{S}} \vec{\Lambda}(\vec{r}) \cdot \left(\hat{S}_+(\vec{r}) + \hat{S}_-(\vec{r})\right) \cdot (\vec{v}_+(\vec{r}) - \vec{v}_-(\vec{r}))\,; \quad (46)$$

$$H[\Gamma, \mathcal{S}; \hat{W}] = \int_{\vec{r}_1,\vec{r}_2\in\mathcal{S}} \frac{d\Gamma(\vec{r}_1)\wedge d\vec{r}_1 \cdot d\Gamma(\vec{r}_1)\wedge d\vec{r}_1}{8\pi|\vec{r}_1 - \vec{r}_2|} + \int_{\vec{r}\in\mathcal{S}} \Gamma(\vec{r}) e_{\alpha\beta\gamma} dr_\beta \wedge dr_\gamma W_{\alpha\mu} r_\mu \quad (47)$$

Minimization by $\Gamma_\pm$ produces the first *CVS* equation (vanishing normal velocity) on each side, and minimization by the Lagrange multiplier $\vec{\Lambda}$ produces the second *CVS* equation.

CVS as a variational problem

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Each $S_\pm(\vec{r}) = (\partial_\alpha \partial_\beta \Phi_\pm)$ and $\vec{v}_\pm = (\partial_\alpha \Phi_\pm)$ represent a linear functional of $\Gamma_\pm$ also depending upon constant tensor $\hat{W}_\pm$.

Therefore, at fixed $\vec{\Lambda}(\vec{r})$ our Hamiltonian is a quadratic functional of $\Gamma_\pm$. Minimization of this quadratic functional represents a linear part of our problem.

Note that all the unknown functions are two dimensional, rather than three-dimensional fields as velocity. This dimensional reduction is a remarkable property of the vortex sheet dynamics

The nonlinear part would be a subsequent minimization by $\vec{\Lambda}, \vec{R}$. In the next talk we present exact (singular) solution of this problem for cylindrical geometry.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

What is the physical origin of the constant background strain $\hat{W}_+ = \hat{S}(\infty)$ which we used in our solution?

Traditionally, the ad hoc Gaussian random forces are added to the Navier-Stokes equation to simulate the effects of the unknown inner randomness.

In our theory, the random forces come from many remote vortex structures, contributing to the background velocity field via the Biot-Savart law.

These forces are not arbitrary; they are rather self-consistent, like a mean-field in ordinary statistical mechanics.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Let us assume that the space is occupied by some localized *CVS* structures far from each other. In other words, let us consider an ideal gas of vortex bubbles.

In such an ideal gas, we can neglect the collision of these extended particles, but not the long-range effect of the strain they impose on each other.

The Biot-Savart formula for the velocity field induced by the set of remote localized vorticity bubbles B

$$\vec{v}(\vec{r}) = \sum_B \int_B d^3r' \frac{\vec{\omega}(\vec{r}') \times (\vec{r}' - \vec{r})}{4\pi |\vec{r} - \vec{r}'|^3} \tag{48}$$

falls off as $1/r^2$ for each bubble, like an electric field from the charged body.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Note that all vortex structures in our infinite volume contribute to this background velocity field, adding up to a large number of small terms at every point in space.

The spacial positions of bubbles are the zero modes of the Euler+ *CVS* equations. Therefore these bubbles are uniformly distributed in space, leading to **spontaneous stochasticity**.

While the Navier-Stokes equation is nonlinear, this relation between the local strain and contributions from each vortex bubble is **exactly linear**, as it follows from the linear Poisson equation relating velocity to vorticity.

The interaction between bubbles decreases with distance by the power law (the multipole expansion of Coulomb kernel), which justifies the ideal gas picture in the case of sparsely distributed vortex bubbles.

The induced background strain

Stable Vortex Surfaces

Hamiltonian Dynamics of Vortex Surfaces

Attractor

Zero Modes

Irreversibility

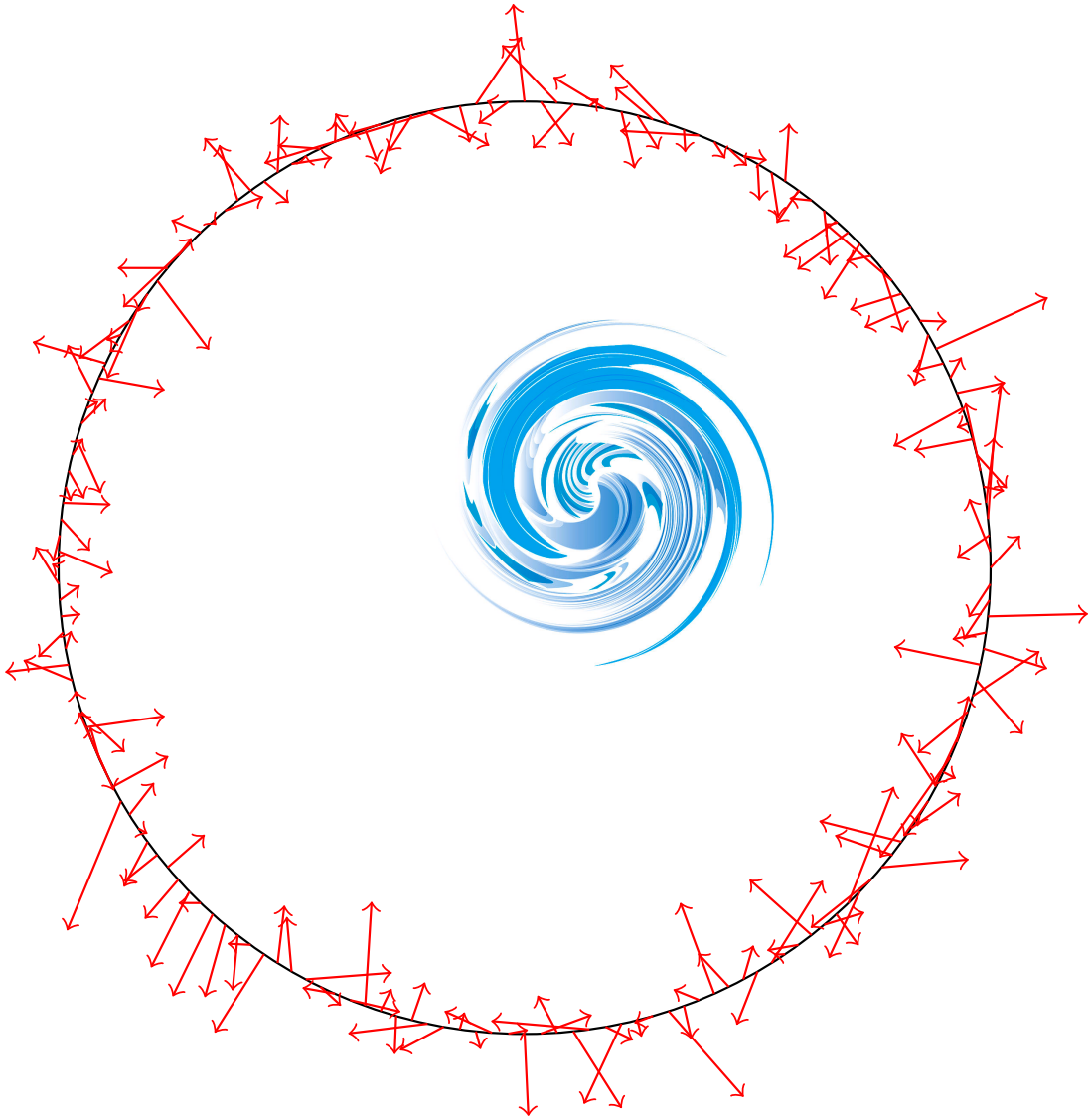
Enstrophy

Stationary Vortex Sheets and Double-layer potentials

CVS as a variational problem

The induced background strain

Energy dissipation and its distribution



The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

This picture symbolizes the vortex bubble under consideration (blue vortex symbol) surrounded by other remote bubbles on a large sphere (orange arrows). The arrows symbolize the directions of these remote bubbles, which follow the local strain main axis and point in random directions.

If there are many such bubbles distributed in space with small but finite density, we would have the "night sky paradox." The bubbles spread on the far away sphere will compensate the inverse distance squared for a divergent distribution like $\int R^2 dR/R^2$.

This estimate is, of course, wrong, as the velocity contributions from various bubbles are uncorrelated, so there is no coherent mean velocity.

Moreover, a Galilean transformation would remove the finite background velocity, so it does not have any physical effects.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

However, with the strain, there is another story.

Strain coming from remote vortex bubbles

$$W_{\alpha\beta}(\vec{r}) = \frac{1}{2} e_{\alpha\mu\gamma}\partial_\beta\partial_\gamma \sum_B \int_B d^3r' \frac{\omega_\mu(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} + \{\alpha \leftrightarrow \beta\}\,; \quad (49)$$

falls off as $1/r^3$, and this time, there could be a mean value $\bar{W}$, coming from a large number of random terms from various bubbles with distribution $R^2 dR/R^3 \sim dR/R$.

The space symmetry arguments and some refined arguments we present in the next section tell us that averaging over the directions of the bubble centers $\vec{R} = \vec{r}' - \vec{r}$ completely cancels this mean value.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The Central Limit Theorem suggests (within our ideal vortex gas model) that such a strain would be a Gaussian tensor variable, satisfying the normal distribution of a symmetric traceless matrix with zero mean

$$dP_\sigma(W) \propto \prod_i dW_{ii} \prod_{i<j} dW_{ij} \delta\left(\sum_i W_{ii}\right) \exp\left(-\frac{\mathrm{tr}\, W^2}{2\sigma^2}\right) \quad (50)$$

The parameter σ is related to the mean mean square of the random matrix. In n dimensional space

$$\frac{(n+2)(n-1)}{2}\sigma^2 = \langle trW^2 \rangle \quad (51)$$

The Gaussian random matrices were studied extensively in physics and mathematics. For example, in [11] the distribution of Gaussian random symmetric matrix (Gaussian Orthogonal Ensemble, $GOE(n)$) is presented.

The induced background strain

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

We achieve the extra condition of zero matrix trace by inserting the delta function of the matrix trace into the invariant measure. This projection preserves the measure's $O(n)$ symmetry as the trace is invariant to orthogonal transformations. We could not find any references for this straightforward extension of the $GOE(n)$ to the space of traceless symmetric matrices.

Separating SO_3 rotations $\Omega \in S_2$, we have the measure for eigenvalues a, b, c:

$$dP_\sigma(W) = \frac{1}{4\pi} d\Omega da db dc \delta(a+b+c) P_\sigma(a,b,c); \tag{52}$$

$$P_\sigma(a,b,c) = \sqrt{\frac{3}{\pi}} \theta(b-a)\theta(c-b)(b-a)(c-a)(c-b) \exp\left(-\frac{a^2+b^2+c^2}{2\sigma^2}\right) \tag{53}$$

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

As we noticed above the total surface dissipation is conserved on *CVS* surfaces.

$$\mathcal{E}_{tot} = \sum_{\mathcal{S}} \mathcal{E}_{\mathcal{S}} = \text{const} \tag{54}$$

Without CVS as a stability condition, the surface dissipation itself would not be an integral of motion. The energy would leak from the vortex surfaces and dissipate in the rest of the volume. Thus, the CVS condition is a necessary part of the vortex sheet turbulence.

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

While the total dissipation is conserved, the individual contributions to this sum from each bubble are not.

The long-term interactions between the vortex bubbles, arising due to the Gaussian fluctuations of the background strain, lead to the statistical distribution of the energy dissipation of an individual bubble.

From analogy with the Gibbs-Boltzmann statistical mechanics, one would expect that the dissipation distribution would come out exponential, with some effective temperature. This hypothesis was put forward in our previous work.

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

However, the interaction between our bubbles is different from that of the Gibbs mechanics. While the background strain is a Gaussian (matrix) variable, the shapes of the bubbles and the corresponding dissipation are not.

These bubbles in our incompressible fluid instantly adjust to the realization of the random background strain. This adjustment is described by our exact solutions of the Euler equations with the *CVS* boundary conditions.

The general formula [9] for the surface dissipation reads

$$\mathcal{E}_{\mathcal{S}} = \frac{\sqrt{\nu}}{2\sqrt{\pi}} \int_{\mathcal{S}} dS \sqrt{-\hat{S}_{nn}} (\Delta \vec{v})^2; \tag{55}$$

where $\int dS$ refers to the surface integral and $\hat{S}_{nn}$ is the normal component of the local strain.

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The dimensional counting using scale invariance of Euler + *CVS* leads to the following formula

$$\frac{\mathcal{E}}{\sqrt{\nu}} = R_0^4 c^{\frac{5}{2}} F(\mu) \tag{56}$$

where $F(\mu)$ is some universal function of the ratio of the two independent eigenvalues

$$\mu = 1 - \frac{c}{a} \tag{57}$$

This function can be found once we know the solution of the *CVS* equations (see the next talk).

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The normalized distribution $W(\zeta)$ for the scaling variable $\zeta = F(\mu)\frac{c^2\sqrt{c}}{\sigma^{\frac{5}{2}}}$ takes the form [12]

$$W(\zeta) = \int_{\frac{3}{2}}^{3} d\mu \frac{2\zeta(3-\mu)\mu(2\mu-3)}{5(\mu-1)^5 F(\mu)} \exp\left(-\frac{((\mu-3)\mu+3)\left(\frac{\zeta}{F(\mu)}\right)^{4/5}}{(\mu-1)^2}\right); \tag{58}$$

with

$$\mathcal{E} = \sqrt{\nu\sigma^5} R_0^4 \zeta \tag{59}$$

Energy dissipation and its distribution

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

This $W(\zeta)$ is a completely universal function. We would measure this function when the distribution of energy dissipation and bubble sizes in numerical or real experiments in the extreme turbulent regime will become available. The size R_0 of the surface remains a free parameter of our theory.

We need some extra restrictions to find the distribution of these perimeters. This extra restriction of the fixed perimeter of the cross-section makes it quite tedious to compare our distribution of the energy dissipation with numerical simulation.

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Let us elaborate on this idea of a dilute gas of vortex bubbles and estimate the strain variance.

Consider a large number of independent vortex bubbles, sparsely distributed in the 3D volume.

The net strain near this surface will come from the Biot-Savart formula, which we expand at large distances

$$W_{\alpha\beta}(\vec{r}) \to \frac{1}{2} e_{\alpha\mu\gamma} \left(\Omega_\mu \partial_\beta \partial_\gamma + \Omega_{\mu\lambda} \partial_\beta \partial_\gamma \partial_\lambda + \dots\right) \frac{1}{4\pi|\vec{r}|} + \{\alpha \leftrightarrow \beta\}; \tag{60}$$

$$\Omega_\mu = \int_B d^3 r' \omega_\mu(\vec{r}'); \tag{61}$$

$$\Omega_{\mu\lambda} = \int_B d^3 r' \omega_\mu(\vec{r}') r'_\lambda \tag{62}$$

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The contribution to the strain from each remote vortex blob will be linearly related to these multipole moments of vorticity.

The leading term is proportional to the net vorticity $\vec{\Omega}$ which vanishes as an integral of closed form $d\vec{\Omega}$ over a closed surface (net flux through the surface without normal vorticity in physics terms).

These vectors and tensors are random variables with zero mean, in addition to the random locations on a sphere, which is why we expect the Central Limit Theorem to apply here).

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The vorticity for each vortex bubble S is given by a surface integral[1], [3]

$$\omega_\nu(\vec{r}) = \int_S d\Gamma \wedge dr'_\nu \delta^3(\vec{r} - \vec{r}'); \tag{63}$$

$$\Omega_\mu = \int_S d\Gamma \wedge dr_\mu = 0; \tag{64}$$

$$\Omega_{\mu\lambda} = \int_S d\Gamma \wedge dr_\mu r_\lambda \tag{65}$$

We get exactly zero when averaged over directions of the position vector $\vec{r}$ of the bubble on the large sphere. We verified that up to the fourth term by symbolic integration [13]. There is, of course, a general reason for these cancellations.

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

The rotational average of the multiple derivative matrix has only one totally symmetric symmetric tensor structure

$$T_{\mu_1,\dots\mu_n} = \left\langle \partial_{\mu_1},\dots\partial_{\mu_n}\frac{1}{|\vec{r}|}\right\rangle_{\vec{r}\in S_2} = \\ = C\left(\delta_{\mu_1\mu_2}\dots\delta_{\mu_{n-1}\mu_n} + \textit{permutations}\right) \qquad (66)$$

However, the contraction over any pair of indices yields zeroes because $\frac{1}{|\vec{r}|}$ satisfies the Laplace equation. Therefore $C = 0$. The number dN of the vortex structures on the large sphere would be estimated as

$$dN = 4\pi\rho(R)R^2dR \qquad (67)$$

where $\rho(R)$ is the distribution of distances between the vortex structures.

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

After some tensor algebra and symbolic angular integration [13] we found the formula for σ with separated averaging over the unit vector on a sphere S_2 and the random tensor W

$$5\sigma^2 = \frac{9}{2\pi^2} \left\langle \Omega_{\alpha\beta}^2 \right\rangle_W 4\pi \int dR \frac{\rho(R)}{R^6}; \tag{68}$$

This distribution is normalized as

$$4\pi \int dR \rho(R) R^2 = 1 \tag{69}$$

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Therefore, our expression involves a mean value of $1/R^8$

$$\left\langle \frac{1}{R^8} \right\rangle = \frac{\int dR\rho(R)R^{-6}}{\int dR\rho(R)R^2}; \tag{70}$$

$$\int dR\rho(R)R^{-6} = \frac{1}{4\pi}\left\langle \frac{1}{R^8} \right\rangle \tag{71}$$

After that, we relate the variance to the mean squared vorticity of each vortex structure and the relative distance distribution of these bubbles.

$$\sigma^2 = \frac{9}{10\pi^2}\left\langle \Omega_{\alpha\beta}^2 \right\rangle_W \left\langle \frac{1}{R^8} \right\rangle \tag{72}$$

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

After integrating over the eigenvalues a, b, c, assuming the scale R_0 fixed, we find the following expression for the variance of strain

$$\sigma^2 = \sigma^2 \text{const} \left\langle R_0^8 \right\rangle \left\langle \frac{1}{R^8} \right\rangle ; \tag{73}$$

$$\bar{R} = \left\langle \frac{1}{R^8} \right\rangle^{-\frac{1}{8}} \tag{74}$$

The variance cancels here and this brings us to the final result for the mean energy dissipation

$$\langle \mathcal{E} \rangle \propto \left\langle R_0^4 \right\rangle \sqrt{\nu \sigma^5}; \tag{75}$$

$$\left\langle R_0^8 \right\rangle \propto \bar{R}^8; \tag{76}$$

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Let us be specific about the geometry scale here: we choose the mean distance between vortex structures $\bar{R}$ as a universal length scale. The sizes of the individual *CVS* surfaces vary and fluctuate, but this $\bar{R}$ is a global parameter of our system so that we can use it as a unit of length.

The mean energy dissipation (for a single *CVS* surface) in this case scales as

$$\langle \mathcal{E} \rangle \propto \bar{R}^4 \sqrt{\nu \sigma^5}; \tag{77}$$

Dilute gas of vortex bubbles in mean-field approximation

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

Thus, we get a scaling relation in the turbulent limit, the same we assumed in previous papers [2], [9]

$$\sigma \sim \left(\frac{\mathcal{E}}{\bar{R}^4}\right)^{\frac{2}{5}} \nu^{-\frac{1}{5}} \tag{78}$$

For the generic solution of the *CVS* equation we find the scale R_0 the same order of magnitude as the mean distance $\bar{R}$ between the bubbles.

However, for the exact cylindrical solution (the tube) we describe in the next talk, there is a small parameter: the ratio of the cross-section perimeter P to the length L of the tube, going to zero as some positive power of viscosity.

For the tubes, therefore, the dilute gas approximation becomes exact in the extreme turbulent limit.

References I

Stable Vortex Surfaces

Hamiltonian Dynamics of Vortex Surfaces

Attractor

Zero Modes

Irreversibility

Enstrophy

Stationary Vortex Sheets and Double-layer potentials

CVS as a variational problem

The induced background strain

Energy dissipation and its distribution

References II

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

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References III

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

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References IV

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

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References V

Stable Vortex Surfaces
Hamiltonian Dynamics of Vortex Surfaces
Attractor
Zero Modes
Irreversibility
Enstrophy
Stationary Vortex Sheets and Double-layer potentials
CVS as a variational problem
The induced background strain
Energy dissipation and its distribution

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References VI

Stable Vortex Surfaces

Hamiltonian Dynamics of Vortex Surfaces

Attractor

Zero Modes

Irreversibility

Enstrophy

Stationary Vortex Sheets and Double-layer potentials

CVS as a variational problem

The induced background strain

Energy dissipation and its distribution

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