PCMI lecture series: from sunflowers to thresholds
Lecture 4 Problems

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You may wish to review the following definitions. Let $\mathcal{F}$ be a family of sets over a universe U, and let $p \in (0, 1)$.

- $\mu_p(\mathcal{F}) \coloneqq \Pr_{W \sim \text{Bin}(U,p)}[\exists S \in \mathcal{F}, S \subset W]$,
- $p_C(\mathcal{F})$ is the minimal p such that $\mu_p(\mathcal{F}) \geq 1/2$,
- $p_E^*(\mathcal{F})$ is the value of p for which $\sum_{S \in \mathcal{F}} p^{|S|} = 1/2$,
- $p_E(\mathcal{F}) \coloneqq \max\{p_E^*(\mathcal{G}) : \mathcal{G} \text{ is a cover of } \mathcal{F}\}$.

1. Verify that the naive expectation thresholds for the family of 4-cliques $\mathcal{F}_{K_4}$ and the family of unions of Hamiltonian cycles and 4-cliques $\mathcal{F}_{\text{Ham},K_4}$ are

$$p_E^*(\mathcal{F}_{K_4}) \approx n^{-2/3} \quad \text{and} \quad p_E^*(\mathcal{F}_{\text{Ham},K_4}) \approx 1/n.$$

2. Prove that the expectation threshold of the family of Hamiltonian cycles $p_E(\mathcal{F}_{\text{Ham}}) = \Theta(1/n)$.

3. Prove that its critical probability $p_C(\mathcal{F}_{\text{Ham}}) = \Theta\left(\frac{\log n}{n}\right)$.

4. Prove that the spread lemma (restated below) follows from the Park-Pham theorem (formerly the Kahn-Kalai conjecture), which states for any family $\mathcal{F}$ of sets of size at most n, we have $p_C(\mathcal{F}) \leq O(p_E(\mathcal{F}) \cdot \log n)$.

Lemma 0.1 (Spread lemma)**.** *Let $\mathcal{F}$ be a family of n-sets defined over a universe U. Let $p \in (0, 1)$ and let $W \sim Bin(U, p)$. Assume that $\mathcal{F}$ is k-spread, where $k = cp^{-1} \log n$ for a large enough absolute constant c. Then*

$$\Pr_W[\exists S \in \mathcal{F}, S \subset W] \geq 1/2.$$

5. In lecture we sketched a proof of the following lemma. Give a complete proof of it.

Lemma 0.2. *Let $p \in (0, 1)$, $q = cp$ for a large enough constant c. Let $\mathcal{F}$ be a family of sets of size $\leq n$ defined over a universe U. Let $V \sim Bin(U, q)$. Then*

$$\mathbb{E}_V \left[\sum_{T \in \mathcal{F}^{large}(V)} p^{|T|} \right] \leq 100^{-n}.$$

6. If you finish with additional time, we encourage you to revisit any problems from previous days that you have not yet had time to consider.