PCMI lecture series: from sunflowers to thresholds
Lecture 1 Problems

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1. Show that there is an infinite family of sets not containing a 3-sunflower.

2. Recall that $SF(n,r)$ denotes the largest size of a family of n-sets that does not contain an r-sunflower. Define the similar quantity $SF^{\leq}(n,r)$ to be the largest size of a family of sets of size *at most* n that does not contain an r-sunflower. Prove that $SF^{\leq}(n,r) = SF(n,r)$.

3. Recall that $ES(N,r)$ denotes the largest size of a family of sets of $[N]$ which does not contain an r-sunflower. Show that $ES(r-1,r) = 2^{r-1} - 1$.

4. In today's lecture we saw that the Erdős-Rado sunflower conjecture implies the Erdős-Szemerédi sunflower conjecture. Show that if we instead apply the sunflower lemma in that proof, then it proves the weaker upper bound

$$ES(N,r) \leq 2^{N - c\sqrt{N}}$$

for some $c = c(r)$.

5. (This question is intended to only be considered if you have completed the previous ones with extra time.) A *cap set* is three points $x, y, z \in \mathbb{F}_3^n$ such that $x + y = 2z$. For many years, it was a prominent open problem in additive combinatorics whether every subset of $\mathbb{F}_3^n$ that does not contain a cap set has size at most $(3-\varepsilon)^n$ for some constant $\varepsilon > 0$. In 2016, Ellenberg and Gijswijt used the polynomial method techniques of Croot, Lev, and Pach to confirm this conjecture. We will show that their solution implies the 3 petal case of the Erdős-Szemerédi sunflower conjecture.

A) Define a 3-sunflower over $\mathbb{F}_3^n$ to be a triple of points x, y, z such that for every $i \in [n]$, either $x_i = y_i = z_i$ or x_i, y_i, z_i are all distinct. Prove that x, y, z form a 3-sunflower if and only if they are a cap set.

B) Let 2^X denote the powerset of a finite set X. Consider the bijection $\phi : 2^{[2n]} \to \{0,1,2,3\}^n$ defined by mapping each pair of consecutive elements in a set's indicator vector to an element of $\{0,1,2,3\}$ according to

$$(0,0) \to 0, \quad (0,1) \to 1, \quad (1,0) \to 2, \quad (1,1) \to 3.$$

Let $\mathcal{F}$ be a set family on the universe $[2n]$. For each $T \subset [n]$, define

$$\mathcal{F}^{(T)} := \{S \in \mathcal{F} : \phi(S)_i = 3 \text{ if and only if } i \in T\}.$$

Observe that we can view $\mathcal{F}^{(T)} \subset \mathbb{F}_3^{n-|T|}$.

Prove that if $\mathcal{F}$ does not contain a 3-sunflower, then $\mathcal{F}^{(T)}$ does not contain a 3-sunflower (over $\mathbb{F}_3^{n-|T|}$).

C) Complete the proof of the Erdős-Szemerédi 3-sunflower conjecture (using known bounds on the cap set problem).

6. (This question is intended to only be considered if you have completed the previous ones with extra time.) Some of you may already be familiar with the proof of the cap set conjecture. For those of you who are not, here is a nice exposition by Grochow, although it will certainly require more time to parse than we have for this problem session.

Apply the polynomial method directly to the 3-sunflower setting to obtain stronger bounds than are given through the approach in the previous problem.