Isadore Singer's Work On Analytic Torsion

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April 6, 2021

I will be talking primarily about just one aspect of Is Singer's many contributions to mathematics and physics, namely his work with Daniel Ray on "analytic torsion":

D. B. Ray and I. M. Singer, "R-Torsion and The Laplacian On Riemannian Manifolds," Adv. Math. **7** (1971) 145-210

D. B. Ray and I. M. Singer, "Analytic Torsion For Complex Manifolds," Ann. Math. **98** (1973) 154-77.

I will also say something about the connection of the Ray-Singer work with physics. This has multiple facets:

- zeta functions and determinants
- torsion and quantum field theory as first perceived in

  A. S. Schwarz, "The Partition Function Of A Degenerate Functional," Comm. Math. Phys. **67** (1979) 1
- torsion for complex manifolds, applications in string theory
- determinants and anomalies, interpretation by Atiyah and Singer
- torsion and volumes of moduli spaces

In the last part of the talk, I will explain a generalization of parts of the story that started here

V. G. Turaev and O. Ya. Viro, "State Sum Invariants of 3-Manifolds and Quantum 6j Symbols," Topology **31** (1992)

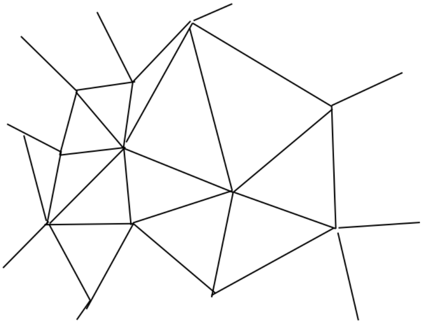
and has many repercussions in contemporary condensed matter physics.

Though I won't try to systematically explain the rest of Singer's work in mathematics and/or physics, which would be well-nigh impossible in a single lecture, various other aspects of Singer's work will play a role today, notably:

- the Atiyah-Singer index theorem
- the Atiyah-Patodi-Singer η invariant
- the topological interpretation of "anomalies" by Atiyah and Singer

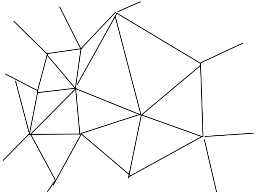
The original "torsion" was the combinatorial torsion introduced by Kurt Reidemeister in 1935. It was historically important because it was the first invariant that could distinguish different manifolds that are homotopy equivalent, for example it could completely classify three-dimensional lens spaces.

One starts with a manifold M described by a simplicial complex, for example a triangulated two-manifold



The manifold M may also be endowed with a flat vector bundle $E \to M$. Following Ray and Singer, I will assume this flat bundle to be unitary, though it is possible to modify the definitions to remove this assumption.

Ray and Singer begin their paper by reviewing the original definition of Reidemeister torsion, but then they explain a variant of that definition that motivated their work. This is as follows. For each q-simplex e_q in the complex



we define E_q to be the space of covariantly constant sections of E over e_q, and $C_q = \oplus_{e_q} E_q$. Then we have a boundary operator

$$\partial : C_q \to C_{q-1}$$

which restricts a covariantly constant section from any e_q to its boundary.

In the usual way, we have $\partial^2 = 0$, and therefore we can define homology groups of ∂, which are the most basic invariants in this situation. However, Reidemeister torsion, which captures information not contained in the homology groups, can be defined as follows. Assuming for simplicity that $E \to M$ is a unitary flat bundle, C_q is a Hilbert space in a natural way, so we can define the adjoint $\partial^\dagger : C_q \to C_{q+1}$ and then we can define a "Laplacian"

$$\Delta = \partial^\dagger \partial + \partial \partial^\dagger .$$

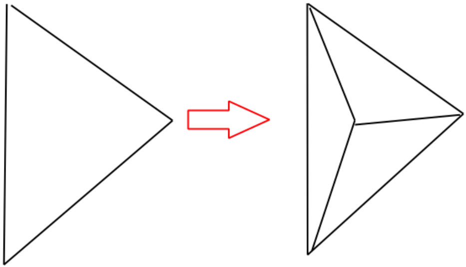
Δ maps C_q to itself for each q, so we define Δ_q to be the restriction of Δ to C_q.

Reidemeister torsion is most simply described if the homology or cohomology groups of M with values in E all vanish (E is "acyclic"), in which case the torsion is simply a number. In that situation, Ray and Singer show that the original definition of the Reidemeister torsion $\tau(E)$ of the flat bundle E, is equivalent to

$$\log \tau(E) = \frac{1}{2} \sum_{q=0}^{N=\dim M} (-1)^{q+1} q \log \det \Delta_q.$$

As I remarked already, this is not quite the original definition of the torsion, but Ray and Singer show it is equivalent.

The key immediate statement about the torsion is that $\tau(E)$ does not depend on the triangulation (simplicial complex) used to compute it, so it is an invariant of the flat bundle $E \to M$. The main step in proving this is to show invariance under subdivision



If the homology groups are nonzero, one has to replace $\det \Delta_q$ (which vanishes because of a nontrivial kernel) with $\det' \Delta_q$, the product of the nonzero eigenvalues of Δ_q. The same formula with det replaced by $\det'$ is still used to define $\tau(E)$, and now the claim is that $\tau(E)$ is invariant if it is interpreted, not as a number, but as a metric or measure on what in modern language would be interpreted as a determinant line bundle.

The idea of Ray and Singer was to make a similar construction in Riemannian geometry. In other words, instead of picking a triangulation of M, they pick a Riemannian metric g on M. Then, letting d denote the exterior derivative acting on E-valued differential forms on M, they define its adjoint $\mathrm{d}^\dagger$ and the corresponding Laplacian $\Delta = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$ and its restriction Δ_q to q-forms. Then assuming that E is acyclic, so the operator Δ_q has trivial kernel, they want to define a "determinant" Δ_q and then to define "analytic torsion" by imitating the formula for Reidemeister torsion:

$$\log T(E) = \frac{1}{2} \sum_{q=0}^{N=\dim M} (-1)^{q+1} q \log \det \Delta_q.$$

One immediate obstacle is to explain what should be meant by the determinant $\det \Delta_q$ of a self-adjoint elliptic operator such as Δ_q. Naively the determinant is the product of the eigenvalues λ_i of Δ_q:

$$\det \Delta_q \stackrel{?}{=} \prod_{i=1}^{\infty} \lambda_i.$$

Ray and Singer had the very nice idea of interpreting this formula via zeta functions and heat kernels.

The *zeta function* of Δ_q is defined as

$$\zeta_q(s) = \sum_i \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{Tr}\, \exp(-t\Delta_q).$$

This converges if $\mathrm{Re}\, s$ is large enough. Its analytic continuation beyond the region of convergence can be analyzed using general results about the small t behavior of the heat kernel $\langle x|e^{-t\Delta_q}|x\rangle$. Schematically with $N = \dim M$ the heat kernel has an asymptotic expansion for small t:

$$\langle x|e^{-t\Delta_q}|x\rangle = \frac{1}{(4\pi t)^{N/2}}(1 + C_0 R(x)t + C_1 \Delta R(x)t^2 + \cdots)$$

Each term in the expansion leads to a pole in $\zeta_q(s)$ at a particular real value of s; higher terms in the expansion give poles at more negative values of s. If N is odd, which for most applications is the main case in the study of the torsion, all the poles are at half-integer values of s and $\zeta_q(s)$ is holomorphic at $s = 0$. (If N is even, there are poles at $s = 0$ in general but they cancel out in the the specific combination of operators considered in the defintion of $T(E)$.)

The formula

$$\zeta(s) = \sum_i \lambda_i^{-s}$$

implies that for a finite-dimensional, positive self-adjoint operator W, the determinant $\det W = \prod_i \lambda_i = \exp(\sum_i \log \lambda_i)$ can be defined as

$$\det W = \exp(-\zeta'(0)).$$

Ray and Singer proposed to use the same definition for det Δ_q, assuming that $\zeta_q(s)$ is holomorphic at $s = 0$:

$$\det\, \Delta_q = \exp(-\zeta_q'(0)).$$

This motivated them to define the "analytic torsion" by

$$\log T(E) = \frac{1}{2} \sum_{q=0}^{N} (-1)^q q \zeta_q'(0).$$

Ray and Singer proved that the analytic torsion is a topological invariant, like the Reidemeister torsion. They also showed that it has many properties in common with the Reidemeister torsion: (1) it is "trivial" on an even-dimensional, oriented manifold (which is why in many applications the torsion is mainly studied on manifolds of odd dimension), (2) in a product $M_1 \times M_2$ where one factor is simply-connected, the analytic torsion $T(E)$ behaves the same way as the Reidemeister torsion $\tau(E)$, (3) if M' is a finite cover of M, then analytic torsion computed on M' is related to analytic torsion on M in the same way that holds for Reidemeister torsion. They conjectured that the analytic torsion and the Reidemeister torsion were equal, and developed a number of tools that they anticipated would be part of a general proof.

The Ray-Singer conjecture was proved a few years later:

J. Cheeger, “Analytic Torsion and Reidemeister Torsion,” PNAS **74** (1977) 2651-4, “Analytic Torsion and the Heat Equation,” Ann. Math. **109** (1979) 259-322

W. Müller, “Analytic Torsion and R Torsion Of Riemannian Manifolds,” Adv. Math. **28** (1978) 233-305.

Before going on, I want to stress that the analysis by Ray and Singer is not restricted to the case that Δ_q has a nontrivial kernel. In general, when the kernel is nontrivial, by using $\det'$ instead of det (in other words by using a modified ζ function defined with only the nonzero eigenvalues), they show that the analytic torsion $T(E)$ is a topological invariant if we interpret it as a measure on (in modern language) a determinant line bundle rather than as a number. This generalization was important for later developments.

In a physicist's language, Ray and Singer took the continuum limit of the combinatorial definition of torsion.

Actually, the proof that Ray and Singer give of the topological invariance of the analytic torsion was based on some elegant manipulations that were reinterpreted a few years later by Albert Schwarz. I will say a word on this when I get to Schwarz's work.

In their second paper, Ray and Singer observed that the $\bar\partial$ operator on a Kahler manifold X has all the formal properties that they had used for the exterior derivative d on a general Riemannian manifold. To generalize their formulation slightly, one can consider the $\bar\partial$ operator acting on $(0,1)$-forms valued in a holomorphic vector bundle $E \to X$. (The case they consider is that E is the bundle of $(p,0)$-forms on X, for some p, tensored with a flat unitary vector bundle over X.) Then they consider the corresponding Laplacian $\Delta = \bar\partial^\dagger\bar\partial + \bar\partial\bar\partial^\dagger$, its restriction Δ_q to $(0,q)$-forms, and the determinant $\det \Delta_q = \exp(-\zeta_q'(0))$. Formally imitating the definition of the Reidemeister torsion, they define the torsion of a holomorphic vector bundle E by

$$\log T(E) = \frac{1}{2}\sum_{q=0}^{N}(-1)^q q\zeta_q'(E).$$

Again, this is the definition if the sheaf cohomology $H^q(X, E)$ vanishes. In this case, Ray and Singer show that the torsion depends only on the complex structure of X and E, and not on the Kahler metric of X (which was used in defining $\bar{\partial}^\dagger$ and Δ). More generally, they define $T(E)$ using $\det' \Delta_q$ (defined as the regularized product of the nonzero eigenvalues). If E is the bundle of $(p, 0)$-forms (for some p) tensored with a flat vector bundle, they show that the analytic torsion $T(E)$ depends only on the Kahler class of X, not on the detailed Kahler metric. For a general holomorphic vector bundle $E \to X$, as considered by later authors, there is a somewhat similar but more elaborate story.

For analytic torsion of a complex manifold, there is no combinatorial version for Ray and Singer to compare to. They explored their definition by computing the analytic torsion of the $(p, 0)$-forms on a Riemann surface X (here $p = 0$ or 1), valued in a flat line bundle $\mathcal{L} \to X$. They showed that the result involves functions of number theoretic interest. For X of genus 1, they computed explicitly and expressed the result in terms of theta functions. For X a hyperbolic surface of genus greater than 1, they related their result to the Selberg trace formula. (Roughly, the Selberg trace formula expresses in terms of a sum over closed geodesics the ζ-function regularized determinant of a slightly more general operator, $\Delta + z(z + 1)$, for a constant z.)

Now I am going to turn to explaining the influence that the Ray-Singer work has had in physics. Recall the rough table of contents:

- zeta functions and determinants
- torsion and quantum field theory as first perceived by A. S. Schwarz
- torsion for complex manifolds, applications in string theory
- determinants and anomalies, interpretation by Atiyah and Singer
- torsion and volumes of moduli spaces
- and last, a twist on part of the story that started with the work of Turaev and Viro (1991) and has many repercussions in modern condensed matter physics.

The first impact of the Ray-Singer work on physics was just that their method of using zeta functions to define regularized determinants was useful. Physicists had known since the work of Richard Feynman and Julian Schwinger around 1950 that regularized determinants of differential operators play an important role in a semiclassical approximation to quantum mechanics. However, the widely used methods of defining these determinants were ineffective and/or inefficient in curved spacetimes. Ray and Singer of course had been working on a curved manifold since the beginning and the ζ function method of defining determinants was very effective in their work. Within a few years, physicists studying quantum field theory in curved spacetimes were using ζ function definition of determinants. The first published reference was apparently by Stuart Dowker and Raymond Critchley in Phys. Rev. **D13** (1976) 3324-32. (They cite earlier work by Phil Candelas and Derek Raine.) There was an influential paper by Hawking:
S. W. Hawking "Zeta Function Regularization Of Path Integrals in Curved Spacetime," Commun. Math. Phys. **55** (1977) 133-48
followed by work by Gary Gibbons and others.

The next development relating analytic torsion to physics was by Albert Schwarz in 1977. Let us write a formula for the analytic torsion rather than its logarithm:

$$T(E) = \prod_{q=0}^{n} (\det{}' \Delta_q)^{-(-1)^q q/2}.$$

In other words, the torsion is a product of determinants of the operators Δ_q for different q, raised to various positive and negative half-integral powers. Such expressions were familiar in physics. The best-known case was simply that the partition function of $\mathrm{U}(1)$ gauge theory on a manifold is

$$\frac{\det' \Delta_0}{(\det' \Delta_1)^{1/2}}.$$

The denominator is the path integral of the gauge field (in a suitable gauge) and the numerator is the path integral of the ghosts, introduced in their earliest version by Feynman. This is similar to the formula for the torsion so one can ask if there is some theory somewhat similar to ordinary $\mathrm{U}(1)$ gauge theory that leads to the torsion. Schwarz showed that there is such a theory.

For simplicity, I will explain Schwarz's idea in the case of $N = 3$ dimensions, which is the first case in which the torsion is an essentially new topological invariant. Let $E \to M$ be a flat bundle described by a flat connection R, with corresponding gauge-covariant exterior derivative $\mathrm{d}_R = \mathrm{d} + [R, \cdot]$. Let A, B be 1-forms on M valued respectively in E and in the dual bundle $E^\vee$. Then Schwarz considered the quadratic action

$$I = \int_M (B, \wedge \mathrm{d}_R A).$$

The corresponding path integral

$$\int DA, DB \ \exp\left(-\int_M (B, \mathrm{d}_R A)\right)$$

is Gaussian, so it can be expressed in terms of determinants. Via standard Faddeev-Popov gauge fixing, Schwarz showed that the appropriate product of determinants is

$$\frac{(\det' \Delta_0)^{3/2}}{(\det' \Delta_1)^{1/2}}.$$

(Note that this is the same that we would have in an ordinary $\mathrm{U}(1)$ gauge theory, except that the exponent in the numerator is $3/2$ instead of 1.)

In 3 dimensions, Poincaré duality gives $\det' \Delta_3 = \det' \Delta_0$, $\det' \Delta_2 = \det' \Delta_1$, so $T(E) = \prod_{q=0}^{3} (\det' \Delta_q)^{-(-1)^q q/2}$, which is the analytic torsion as defined by Ray and Singer, reduces to $(\det' \Delta_0)^{3/2} / (\det' \Delta_1)^{1/2}$, which comes from Schwarz's calculation.

So the theory considered by Schwarz, with

$$I = \int_M (B, \wedge \mathrm{d}_R A),$$

has the property that its parttition function is the analytic torsion of Ray and Singer, and so in particular is a topological invariant. Why did this happen? The point is that the action I can be defined on any smooth manifold M. (M does not even have to be oriented, if B is viewed as a 1-form twisted by the orientation bundle of M.) In particular, no Riemannian metric is required. However, to quantize the theory in a way that leads to the formula involving determinants, one has to first fix a gauge and this gauge choice does require a choice of Riemannian metric on M. Then the Ray-Singer theorem that the torsion does not depend on the metric of spacetime is a special case of the statement that the partition function of the theory is independent of the gauge. A physicist looking at the matter today would probably use the machinery of BRST quantization to find the identity that implies that the torsion does not depend on the metric. Ray and Singer had found this identity by hand.

Schwarz's work was not limited to the case of three dimensions. In any dimension N, he similarly considered the action

$$I = \int_M (B, \mathrm{d}_R A),$$

where A is a p-form valued in E (for some p), and B is an $N - p - 1$-form valued in the dual bundle $E^\vee$. Since at least one of A, B is a form of degree greater than 1, a generalization of the standard Faddeev-Popov or BRST gauge fixing is required, and Schwarz provided this (anticipating some aspects of the modern BV approach to quantization). Schwarz showed that, for any p, the partition function of this theory is the analytic torsion $T(E)$. The fact that the partition function of this theory is independent of p is somewhat puzzling to me, even today.

To summarize part of this more briefly, Schwarz's explanation of the topological invariance of the analytic torsion was that the torsion comes by quantizing a theory

$$I = \int_M (B, \wedge \mathrm{d}_R A)$$

that can be defined on any smooth manifold M with no additional structure. This is a slightly formal statement. To turn it into a real argument, one needs to analyze the quantization carefully enough to show that there is no possible "anomaly." That step is actually not difficult. It is reasonable to view Schwarz's paper as the first paper on topological field theory from a physics perspective. Of course, the papers of Reidemeister and of Ray and Singer were important precursors, from a math perspective. I was aware of Schwarz's paper, because Sidney Coleman pointed it out to me soon after it appeared.

A decade later, trying to understand the Jones polynomial in quantum field theory, I considered a theory on an oriented three-manifold M with gauge group G, gauge field A, and action a multiple of the Chern-Simons functional:

$$I = \frac{k}{4\pi}\int_M \mathrm{Tr}\,\left(A\mathrm{d}A + \frac{2}{3}A^3\right).$$

Since this action is not quadratic in A, the path integral is not a simple Gaussian and cannot be expressed just in terms of determinants. However, determinants do arise in a semiclassical approximation. The Euler-Lagrange equation for a critical point of the functional I just says that the curvature $F = \mathrm{d}A + A \wedge A$ should vanish. So let A_0 be a classical solution, that is a flat connection. It is a little easier to consider first the case that the holonomy of A_0 is irreducible (it commutes only with the center of G) and that the classical solution corresponding to A_0 is isolated – it has no moduli. These assumptions are equivalent to saying that the flat bundle E that corresponds to A_0 has the property that the cohomology $H^q(M, \mathrm{ad}(E)) = 0$ for all q, where $\mathrm{ad}(E)$ is the adjoint bundle associated to E. Remember that that is the condition that makes the torsion of $\mathrm{ad}(E)$ a topologically invariant number.

Now we write $A = A_0 + B$, where A_0 is a classical solution and the "quantum fluctuation" B will be small if the Chern-Simons "level" k is large. The path integral over B is (in the large k limit)

$$\int DB \exp(\mathrm{i}I(A_0 + B))$$

where

$$I(A_0 + B) = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A_0 \mathrm{d}A_0 + \frac{2}{3}A_0^3\right) + \frac{k}{4\pi}\int_M \mathrm{Tr} B\mathrm{d}_{A_0}B.$$

The first term is just a constant $I(A_0)$, the classical action of the classical solution A_0. This leads to a phase factor $\exp(\mathrm{i}I(A_0))$ in the path integral (for large k, this factor is highly oscillatory as $I(A_0)$ is proportional to k). Let us look at the part of the action that depends on the quantum fluctuation B:

$$I(B) = \frac{k}{4\pi}\int_M \mathrm{Tr}\, B\mathrm{d}_{A_0}B.$$

Let us compare this action

$$I(B) = \frac{k}{4\pi} \int_M \mathrm{Tr}\, B \mathrm{d}_{A_0} B.$$

to the theory studied by Schwarz:

$$I_{\mathrm{Sch}}(A, B) = \int_M (B, \mathrm{d}_R A).$$

There are a few cosmetic differences: there is an inessential extra factor $k/4\pi$, the background flat connection has been called A_0 rather than R, and the pairing $(\ ,\ )$ between a flat bundle and its dual is now called Tr (with the flat bundle being $\mathrm{ad}(E)$). The only important difference is that, relative to the case considered by Schwarz, one now has $A = B$.

Let us write $A \pm B = C_\pm/2$, so $C_+^2 - C_-^2 = AB$. The "Schwarz" path integral is therefore

$$\int DA\,DB\,\exp(\mathrm{i}\int (B, \mathrm{d}_R A))$$

$$= \int DC_+ \exp(\mathrm{i}\int C_+ \mathrm{d}_R C_+) \int DC_- \exp(-\mathrm{i}\int (C_-, \mathrm{d}_R C_-).$$

We see that because of the opposite signs in the exponent, the C_+ and C_- path integrals are complex conjugates of each other. The C_+ path integral is equivalent to the one that we got in Chern-Simons theory (in this 1-loop approximation) with $C_+ \to B$ (and $R \to A_0$) as from Chern-Simons we had

$$\int DB \exp\left(\mathrm{i}\frac{k}{4\pi}\int Tr\, B \mathrm{d}_{A_0} B\right).$$

The conclusion then is that the "Schwarz" or AB path integral is the absolute value squared of the 1-loop path integral of the Chern-Simons theory. Since the AB path integral equals the torsion, the Chern-Simons path integral is equal to the square root of the torsion, times something of modulus 1.

A more careful analysis shows that the factor of modulus 1 is related to another contribution by Singer – it can be expressed in terms of the η-invariant of Atiyah, Patodi, and Singer (1973). The upshot is that in the 1-loop approximation, the contribution of an isolated classical solution A_0 to the path integral is essentially

$$\frac{1}{\#G} \exp(\mathrm{i}I(A_0))\sqrt{T(A_0)}\exp(\mathrm{i}\pi\eta(A_0)/2).$$

(Here $\#G$ is the order of the center of G, which enters when one treats precisely the Fadeev-Popov gauge fixing. For a careful discussion of this formula and comparison to exact calculations by other methods, see D. Freed and R. Gompf, Commun. Math. Phys. **141** (1991) 79-117.)

So far we have assumed that the flat connection A_0 is isolated – up to a gauge transformation, it has no deformation that preserves the condition $\mathrm{d}A + A \wedge A = 0$. We also assume that A_0 is irreducible. The combined conditions say that the flat bundle $\mathrm{ad}(E)$ corresponding to A_0 is acyclic, so the torsion $T(A_0)$ is a number.

As I explained earlier, Ray and Singer defined the torsion without such assumptions as a metric on a certain line bundle. In the context of Chern-Simons theory in three dimensions, this has a very nice interpretation. Let us keep the assumption that A_0 is irreducible but drop the assumption that it is isolated. Then A_0 defines a point in a moduli space $\mathcal{M}$ of flat connections over the three-manifold M. In the path integral, we expect to have to integrate over $\mathcal{M}$ (as well as over small quantum fluctuations in directions normal to $\mathcal{M}$). What is the measure that we will integrate over $\mathcal{M}$, in the 1-loop approximation?

The answer is that in this situation, the logic of Ray and Singer shows that $\sqrt{T(A_0)}$ can be interpreted as a measure on $\mathcal{M}$. In other words, the line bundle on which $\sqrt{T(A_0)}$ turns out to be a measure is precisely the determinant of the tangent space to $\mathcal{M}$ at the point corresponding to A_0. Thus what has to be integrated over $\mathcal{M}$ in the 1-loop approximation is precisely what we described before: the contribution of $\mathcal{M}$ to the path integral is

$$\frac{1}{\#G}\int_{\mathcal{M}} \exp(\mathrm{i}I(A_0))\sqrt{T(A_0)}\exp(\mathrm{i}\pi\eta(A_0)/2).$$

All this is the 1-loop approximation to the path integral; there are further corrections involving an asymptotic expansion in powers of $1/k$. Singer became very interested in this expansion. With Scott Axelrod, he established its well-foundedness in a very elegant way, for the case of expanding around an acyclic flat connection. (See their paper J. Diff. Geom. **39** (1994) 173-213).

There is another situation in which the interpretation of the torsion as a measure is extremely useful. This happens in a theory in 2 dimensions that nowadays is usually called BF theory; it is even more similar than 3d Chern-Simons theory to the theory considered by Schwarz. In two dimensions, on a two-manifold Σ, we consider a theory with a gauge G and a connection A on a G-bundle $E \to \Sigma$. As usual the curvature is $F = \mathrm{d}A + A \wedge A$. We also include a field B that is a section of $\mathrm{ad}(E) \otimes \mathrm{or}$, where or (trivial if Σ is orientable) is the orientation bundle of Σ. The action is

$$I = \int_\Sigma \mathrm{Tr} BF.$$

The path integral "localizes" on the moduli space $\mathcal{M}$ of flat connections on E as one can see by an elementary calculation.

To see the localization, we perform the path integral over B first:

$$Z = \frac{1}{\mathrm{vol}} \int DA\, DB \exp(\mathrm{i} \int_{\Sigma} \mathrm{Tr} BF) = \frac{1}{\mathrm{vol}} \int DA \delta(F).$$

When we do the integral over A with the help of the delta function, we get a ratio of determinants which precisely corresponds to the torsion $T(A)$. The result is that the "partition function" Z is the integral of $T(A)$ over the moduli space $\mathcal{M}$:

$$Z = \int_{\mathcal{M}} T(A).$$

This makes sense, because if one follows the logic of Ray and Singer, one finds that in two dimensions, $T(A)$ is a measure on $\mathcal{M}$ (while in three dimensions $\sqrt{T(A)}$ is a measure on $\mathcal{M}$).

If Σ is orientable, the measure that comes from $T(A)$ coincides with the measure on $\mathcal{M}$ that can be defined using its symplectic structure. (Equivalence of these two measures results from the fact that "torsion is trivial on an orientable manifold of even dimension," which is one of the common properties of Reidemeister and analytic torsion that had been analyzed by Ray and Singer.) For unorientable Σ, as far as I know, the measure that comes from $T(A)$ is not equivalent to anything more elementary. (See L. Jeffrey and N.-K. Ho, arXiv:math/0307404, for more on these volumes in the unorientable case.) Whether Σ is orientable or not, the integral $\int_M T(A)$ can be computed quite explicitly, using properties of the torsion that go somewhat beyond when we have time for today.

Now I would like to say something about the influence in string theory of $\bar{\partial}$ version of the Ray-Singer analytic torsion, which I will also call the analytic torsion of a complex manifold. First I will explain how it has been used in physics if we take the question literally, and then I will comment on what other developments it has helped inspire.

It is mainly in string theory that physicists have run into the analytic torsion of a complex manifold. First of all, the "worldsheet" of a string is a complex Riemann surface. The building blocks of string perturbation theory (in the simplest string theory constructions) are products of determinants on the string worldsheet, and these determinants are very closely related to the analytic torsion. These determinants are the examples that Ray and Singer studied by way of illustrating their complex version of analytic torsion.

However, we encounter analytic torsion of a complex manifold of higher dimension when we consider compactifications of string theory. The most studied case is a Calabi-Yau threefold X. In particular, the 1-loop contributions for the B-model with a Calabi-Yau target space involve the analytic torsion. This is true both for closed strings and for open strings. For closed strings, Bershadsky, Cecotti, Ooguri and Vafa (Commun. Math. Phys. **165** 1994) expressed the 1-loop B-model amplitude with target X in terms of the analytic torsion (for the vector bundles that were actually considered by Ray and Singer, namely the bundles of $(p, 0)$-forms for various p).

If one considers the 1-loop B-model for open strings on X, one runs into the analytic torsion for a rather large class of holomorphic vector bundles $E \to X$. The reason for this is that one runs into a holomoprhic version of Chern-Simons theory, a gauge theory of a connection on a smooth vector bundle $E \to X$ such that only the $(0,1)$ part of the connection, which I will call $\mathcal{A}$, is relevant, and the action is a Chern-Simons $(0,3)$-form of $\mathcal{A}$:

$$I = \int_X \Omega \mathrm{Tr} \left(\mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3 \right).$$

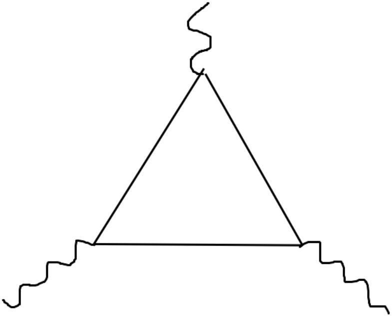
(Here Ω is a holomorphic $(3,0)$ form on X, which has been used in defining the B-model.) Many of the things that I said in the discussion of ordinary Chern-Simons theory have analogs here, with the $\bar{\partial}$ version of analytic torsion playing the role that in ordinary Chern-Simons theory is played by the ordinary analytic torsion.

However, there is another important development that I think was likely inspired in part by the understanding of $\bar{\partial}$ analytic torsion. As background, I have to explain a little physics. First let me recall how the importance of the Atiyah-Singer index theorem first came to be appreciated by physicists. By about 1974, Quantum Chromodynamics – $SU(3)$ gauge theory with quarks, also called QCD – had emerged as a candidate theory of the nuclear force (also called the strong interactions). There was a problem: there was an *almost* perfect match between the symmetries of QCD and the experimentally observed symmetries of the nuclear force, but there was one troubling discrepancy. QCD appeared to have an extra $\mathrm{U}(1)$ symmetry that was not observed in nature.

Then the Yang-Mills instanton was discovered, and in 1976, Gerard 't Hooft (inspired in part by earlier work of C. Rebbi and R. Jackiw) discovered that in the field of an instanton, the kernel of the Dirac operator is not invariant under the troublesome $\mathrm{U}(1)$ symmetry. It was soon explained by Albert Schwarz (in another paper that I learned about from Sidney Coleman, by the way) that what was happening was a manifestation of the Atiyah-Singer index theorem. The properties of the Dirac operator that were leading to the solution of the $\mathrm{U}(1)$ problem were predicted by the index theorem.

This development had a huge impact in physics – hard to exaggerate. I tend to assume that it is part of what got Singer and Atiyah, among others, interested in physics.

Physicists described what was happening as an “anomaly” – an apparent symmetry of the theory was in fact not a valid symmetry because of subtleties introduced in the process of regularizing it. This particular anomaly was called a triangle anomaly because the Feynman diagram that one has to calculate to see the anomaly is a triangle:



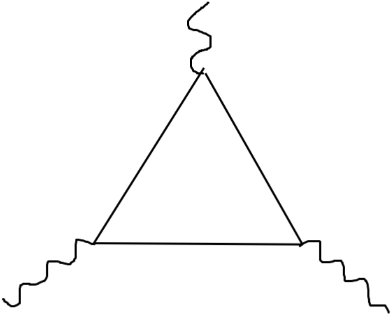
The anomaly in this particular Feynman diagram had been discovered around 1970 by Steve Adler and by Jackiw and John Bell. They had applied it to another important problem (involving the decay of the π^0 elementary particle), but one that had not required a familiarity with instantons or the index theorem.

The same triangle anomaly had yet another important manifestation in physics. This involves "parity" – the symmetry of reflecting a direction in space, exchanging left and right and reversing the orientation of space. It turns out that parity is a symmetry of the nuclear force and of electromagnetism, but it is not a symmetry of the weak interactions. Mathematically, one consequence of this is that when one formulates the Standard Model of particle physics in Euclidean signature, one has to use a Dirac operator $\not{D}$ that is far from being self-adjoint; it maps sections of one vector bundle E to sections of another vector bundle F. Formally then the Euclidean signature path integral of the Standard Model involves a factor $\det \not{D}$ where $\not{D}$ is highly non-self-adjoint.

Does this determinant make sense? It certainly cannot be defined by the ζ-function method of Ray and Singer. To define the zeta function $\sum_i \lambda_i^{-s}$ of an operator, it needs to have eigenvalues, but an operator that maps sections of one bundle E to sections of a different bundle F does not have eigenvalues.

Physicists of the time were certainly not computing determinants of operators via ζ-functions. Instead they were using pertubation theory, by means of which an interesting necessary condition for well-definedness of the determinant had been discovered (by Bouchiat, Iliopoulos, and Maiani, and by Gross and Jackiw).

Formally, the construction of the perturbation theory involved standard physical ideas, but when one implemented these ideas one ran into the same triangle anomaly



by which Adler and Bell-Jackiw had solved the puzzle about π^0 decay. One found that a certain condition on the "quantum numbers" of quarks and leptons had to be satisfied or the fermion determinant would not make sense. This criterion had important implications. For example, the top quark was predicted before it was discovered because without it, the fermion determinant of the Standard Model could not be defined.

But what was the mathematical interpretation of what physicists were doing in these computations? This question was elucidated by Atiyah and Singer in 1984 ("Dirac Operators Coupled To Vector Potentials," PNAS (1984) 2597-2600). The idea (as elaborated later, in part, by Singer's student D. Freed) was the following. When $\not{D}$ maps one vector bundle E to another bundle F, its determinant may not make sense as a number, but it always makes sense as a section of a certain determinant line bundle. Moreover, this determinant line bundle carries a natural hermitian metric and a natural connection, which (in their interpretation) is being constructed in the computations of the physicists. Finally, if this natural connection is flat with trivial global holonomy, then $\det \not{D}$ can be defined as a complex-valued function, as physicists wanted in the Standard Model. So in short, according to Atiyah and Singer, what physicists were computing with the triangle calculation was the curvature of a natural connection on the determinant line bundle.

This interpretation of the triangle anomaly was very illuminating for physicists. It clarified the meaning of a number of important computations by physicists. I believe that in advancing this interpretation of the anomaly, Atiyah and Singer were in part generalizing to differential geometry some things that were more obvious in complex geometry. In complex geometry, the torsion was defined by Ray and Singer as (in modern language) a hermitan metric on a determinant line bundle. In complex geometry, a hermitian metric on a line bundle automatically determines a connection. It is less obvious that the determinant line bundle has a natural connection in differential geometry, but the interpretation of Atiyah and Singer was that it does and that that is what physicists were analyzing.

The practical import of the analysis of anomalies is to learn which Standard Model-like theories – i.e., gauge theories with fermions – are consistent and so could potentially be applicable in describing Nature.

Finally I want to explain a remarkable twist on the equivalence between analytic and combinatorial (Reidemeister) torsion. Let us accept A. Schwarz's interpretation of the torsion as the partition function of a continuum topological field theory. Then the Ray-Singer conjecture means that this continuum topological field theory is equivalent to a theory – the combinatorial torsion – that actually can be defined on a spacetime lattice (a simplicial complex in mathematical language). I was familiar with this interpretation more than thirty years ago when I was working on Chern-Simons gauge theory. I did not technically understand exactly how to write a lattice version of the $\int(B, \mathrm{d}_R A)$ action but I realized that the equivalence of analytic and combinatorial torsion suggested that something like this was possible. Regardless, I assumed that the existence of a combinatorial description depended on the fact that the $\int(B, \mathrm{d}_R A)$ theory is free (quadratic in the fields A, B so that the partition function can be written just in terms of determinants).

Remarkably, V. G. Turaev and O. Ya. Viro discovered (1992) that this is not true. If G is a compact gauge group then the "level k" Chern-Simons theory with action

$$\frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\mathrm{d}A+\frac{2}{3}A^3\right)$$

does not (it is believed) have a combinatorial description. But suppose we consider two copies of this theory, with equal and opposite levels. The action is

$$\frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\mathrm{d}A+\frac{2}{3}A^3\right)-\frac{k}{4\pi}\int_M \mathrm{Tr}\left(B\mathrm{d}B+\frac{2}{3}B^3\right)$$

where A, B are the two gauge fields. Turaev and Viro discovered a description of this model as what a physicist would call a "lattice model," in which the partition function is written in terms of local data defined on simplices and summed over, representing the partition function as an explicit sum over locally-defined data. (It is the locally defined data, not the explicit sum, that is the surprise.)

More recently condensed matter physicists became interested in this phenomenon and discovered lattice representations of many three-dimensional topological field theories (for a small sample of references: Levin and Wen (2004), Kitaev and Kong (2012), Bhardwaj, Gaiotto, and Kapustin, (2016)). The modern understanding is the following: a topological field theory has a lattice representation if and only if it admits a gapped boundary.

Let me explain this criterion in our examples. The Schwarz theory

$$\int_M (B, \mathrm{d}_R A)$$

actually has two possible gapped boundary conditions, namely $A = 0$ with B free and $B = 0$ with A free. So it has a combinatorial counterpart, which is the combinatorial torsion. The Chern-Simons theory with action

$$\frac{k}{4\pi} \int_M \mathrm{Tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right)$$

has no gauge-invariant gapped boundary condition. Instead it has a gapless boundary condition with chiral currents (i.e. a left-moving version of the WZW model) on the boundary. This is important in its physical applications and in the way it is solved. (Absence of gapped boundary was proved by Freed and Teleman, arXiv: 2006.10200.) The doubled Chern-Simons theory with equal and opposite levels

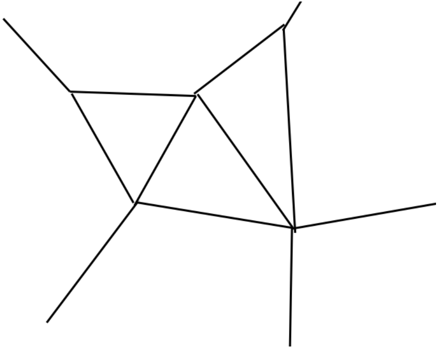
$$\frac{k}{4\pi} \int_M \mathrm{Tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right) - \frac{k}{4\pi} \int_M \mathrm{Tr}\left(B\mathrm{d}B + \frac{2}{3}B^3\right)$$

has a gapped boundary condition $A = B$, so it has the combinatorial description discovered by Turaev and Viro.

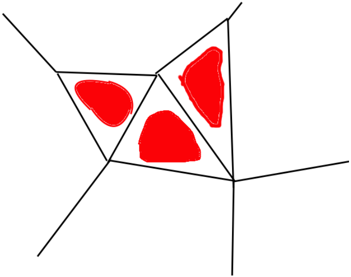
I will only give a sketch of why when a topological field theory has a gapped boundary condition, this leads to a combinatorial description. (I follow Bhardwaj, Gaiotto, and Kapustin.) Consider a topological field theory in dimension D. Suppose we remove an open ball from a D-manifold M.

In a unitary topological field theory, this multiplies the partition function by a universal constant, independent of M.

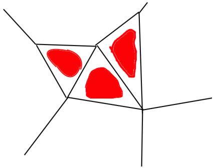
Now consider any triangulated D-manifold, drawn here for $D = 2$:



Remove an open ball from every D-simplex:



What is left



is a slightly thickened version of the $(D-1)$-skeleton of the manifold. What one sees on each $D-1$ simplex S is a $D-1$-dimensional topological field theory (obtained by "compactification" on an interval normal to S) of the original D-dimensional theory. The $D-1$-dimensional theory has a finite set of "vacua." For example, in the Turaev-Viro case, these vacua are the integrable representations of G at level k (the representations of the quantum group G_q at $q = \exp(2\pi \mathrm{i}/(k_h)))$. To calculate the partition function of the system, we can sum over labeling of the $D-1$ simplices by those vacua. Then we have to sum over local data where $D-1$-simplices meet along $D-2$-simplices, and then further data at intersections along $D-3$-simplices, and so on down to the bottom.

The conclusion is a remarkable generalization of the equivalence between analytic and combinatorial torsion that was discovered nearly half a century ago with Ray and Singer. This generalization is of considerable interest in contemporary physics.

To summarize, I reviewed the two famous papers of Ray and Singer on analytic torsion and then I talked about the influence of this work in physics in these areas:

- zeta functions and determinants
- torsion and quantum field theory as first perceived by A. S. Schwarz
- torsion for complex manifolds, applications in string theory
- determinants and anomalies, interpretation by Atiyah and Singer
- torsion and volumes of moduli spaces
- and last, a twist on part of the story that started with the work of Turaev and Viro (1991) and has many repercussions in modern condensed matter physics.