ALTERNATIVE CONSTRUCTIONS OF WEAK SOLUTIONS OF THE EULER EQUATIONS

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The proposed lecture minicourse of 3 lectures is devoted to the alternative constructions of weak solutions of the Euler equations. The existing, most successful constructions provide the full solution of the problem posed by Onsager: prove that for any $\alpha < \frac{1}{3}$ there exists a weak solution $u(x,t) \in C^\alpha$ of 3-d Euler equations such that its kinetic energy $E = \frac{1}{2}||u||^2_{L^2}$ decreases, i.e. there is energy dissipation in absense of a true (molecular) viscosity. However, the solutions constructed are excessively "flexible": the energy can not only decrease, but also it can increase, and moreover, it can be made equal to any prescribed nonnegative function $e(t)$. In particular, it is possible to construct such solution starting from zero (the Scheffer's phenomenon in C^α). This can be related to the fact that the constructed weak solutions are time-reversible, i.e. if $u(x,t)$ is such solution, then $-u(x,-t)$ is a solution, too. It appears that this is a common feature of all solutions obtained by the convex integration.

This makes it necessary to consider alternative approaches to construction of dissipative weak solutions which take into account more "physical" aspects of the problem. In the proposed lectures I'd like to describe several new ideas in this direction. They all are based on some generalizations of the classical concepts of mechanics like the configuration space, the variational principle, and the D'Alembert Principle.

Here is a more detailed description of the proposed topics.

1. Variational method. Let A, B be two fluid configurations, defined by diffeomorphisms $f, g \in SDiff(M)$ where M is a bounded 2-d domain, or a compact Riemannian surface. Consider the following problem: find a path $h_t \in SDiff(M)$ connecting f and g such that its L^2-action $\int_0^1 ||h_t||^2_{L^2} dt$ were minimal among all the paths connecting f and g. Classical solution of this problem generally does not exist; however, there exists a weak Lagrangian solution. This solution is a Generalized Flow (GF) in the sense of Brenier; however, it possesses a meaningful velocity field which is a weak solution of 2-d Euler equations. The key to this result is a careful topological analysis of the GFs which are regarded as continual analogue of braids. Further development of this theory yields a construction of a weak solution of constant energy which weakly tends to zero as $t \to t_0$ which is a new example of the Scheffer's effect, since it can be continued by zero past $t = t_0$, and form a weak solution with decreasing energy (in this case in a jump). This approach also provides a new insight in two recent breakthroughs: construction of a stationary (time-independent) weak solution with compact support, and construction of a weak solution in a bounded domain with arbitrarily knotted flow lines.

2. Variational construction of weak solution in 3-d using the extended configuration space. In the classical mechanics there is no way to extend the Least Action Principle to the dissipative processes like inelastic collision of material particles. However, it is possible to restore this possibility by adding some new, "invisible" degrees of freedom where a part of the energy can escape. The extended configuration space has a highly nonlinear and nonsmooth structure; it is a sort of "building" $\mathbf{B}$, a collection of planes in a higher-dimensional Euclidean space joined together along their intersection. The Least Action Principle now applies if we put A and B, the initial and the final points of the shortest path, on the building $\mathbf{B}$, and then consider its projection on the physical space. We get the evolution of a system of sticky particles in the Euclidean space.

This approach can be applied to the variational description of a Generalized Flow with sticky particles, so that the energy is dissipated upon their collision. I construct the corresponding extended configuration space, the "building" $\mathbf{B}$. The flow construction requires method in the spirit of convex integration; the resulting weak solution with monotone decreasing energy looks a more realistic model of strong turbulence.

3. Turbulent flow as a motion on a rough surface and the D'Alembert Principle. Let M be a bounded domain in $\mathbb{R}^n$, and $D = SDiff(M)$ the group of volume-preserving diffeomorphisms of M. There is a natural embedding of D into the Euclidean space $X = L^2(M, \mathbb{R}^n)$ which is an isometry (w.r.t. the L^2 metric on M); in what follows we regard D as embedded in X. The classical Euler equations can be formally regarded as the motion equation of a material point along the "surface" D following the D'Alembert Principle: the acceleration at every point of the particle trajectory is orthogonal to the tanvent space to D at the point.

However, the "surface" D is far from a smooth surface; in fact, it is pretty rough. Its degree of roughness is comparable with the graph of a function $\phi \in C^\alpha$ where $\alpha \approx \frac{2}{7}$. Thus, we are forced to study the motion of a material point on a rough surface. It should be noted that at every point $f \in D$ of the "surface" D there exists a substitute of a tangent plane, Y_f; these planes form a (non-smooth and non-integrable) distribution on D. So, we have to define the motion f_t of a material point along a nonsmooth surface D, subject to the nonsmooth and nonholonomic constraint $\dot{f}_t \in Y_{f_t}$. This problem is solved using the tools of Nonstandard Analysis. I'm going to sketch the construction.

It should be said that the energy loss in the turbulent flows is associated with the nonsmoothness of the flow (the dissipation rate is defined by the Duchon-Robert function which is zero for regular flows). However, the true cause of dissipation is the nonsmoothness of the underlying configuration space D! Hence, this effect can be observed in much simpler cases of condtrained mechanical systems with nonsmooth constraints. And in fact, I have found 1-dimensinal systems, namely the motion on some nonsmooth curves in the plane where this effect is present. These are the curves such that the angle formed by the tangent vector and the x-axis, as a function of the length parameter, is (a typical realization of) the Wiener process.

I'll try to tell as much as possible of the things sketched above.