Report for the IAS Summer Collaborators Program: Geometry of moduli spaces of holomorphic dynamical systems

Fabrizio Bianchi (ORCID 0000-0002-6720-3211) and Yan Mary He (ORCID 0000-0002-0729-7711)

1. Overview

The Teichmüller space $T(S)$ of a closed surface S of genus at least 2 plays a fundamental role in modern mathematics. There are a number of natural metrics on $T(S)$ defined from different perspectives, e.g., the Teichmuller metric, the Weil-Petersson metric and the Thurston metric. The Weil-Petersson metric on $T(S)$ can be reconstructed via thermodynamic formalism, and is a constant multiple of the so-called *pressure metric*. Moreover, pressure metrics have been constructed and studied in various contexts in geometric topology, including but not limited to quasi-Fuchsian spaces of closed surfaces, Teichmüller spaces and quasi-Fuchsian spaces of punctured surfaces, deformations spaces of Anosov representations, cusped Hitchin components, moduli space of metric graphs and Culler-Vogtmann outer spaces.

During our stay at the Institute for Advanced Study (July 19-31, 2025), we studied pressure metrics in the context of *holomorphic dynamics*, and particularly in the setting of holomorphic families of endomorphisms of $\mathbb{CP}^k$, for both $k = 1$ and $k \geq 2$. Our general goal can be summarized as follows:

Show that (apart from trivial exceptions) any stable component of a holomorphic family of endomorphisms of $\mathbb{P}^k$ *carries a natural pressure metric, which parallels those in the contexts mentioned above.*

More specifically, during our stay at the Institute, we have discussed and completed the following papers that we plan to submit for publication soon:

1. Completed the paper entitled *Analyticity of the Hausdorff dimension and metric structures on families of Misiurewicz maps.*
2. Completed the paper entitled *Manhattan curves in complex dynamics and asymptotic correlation of multiplier spectra.*

In what follows, we discuss the contents of these two papers in more details.

2. Analyticity of the Hausdorff dimension and metric structures on families of Misiurewicz maps

Stable components of moduli spaces of rational maps, seen as holomorphic dynamical systems on the Riemann sphere, are the natural counterparts in complex dynamics of Teichmüller spaces of closed surfaces. This correspondence provides both the motivation and the tools to examine both the topology and the geometry of such stable components, in parallel to the theories of Teichmüller spaces.

Ever since McMullen's [McM08] construction of the Weil-Petersson metric on the space of degree $d \geq 2$ Blaschke products in complex dynamics, there has been an extensive study of Weil-Petersson metrics on stable components of moduli spaces. Ivrii [Ivr14] studied the completeness properties of McMullen's metric for degree 2 Blaschke products. Nie and the second participant [HN23] constructed Weil-Petersson metrics on general hyperbolic components in moduli spaces of rational maps. Lee, Park, and the second participant [HLP24; HLP25] studied the degeneracy loci of the Weil-Petersson metric on spaces of quasi-Blaschke products. In our earlier paper [BH24b], we studied the Weil-Petersson metric on stable components of polynomials families with a persistent parabolic point.

Our goal is to extend the theory of Weil-Petersson metrics in complex dynamics to stable components of a *Misiurewicz family* of polynomials (i.e., a family where some critical point is persistently preperiodic to some repelling periodic point). The construction of Weil-Petersson metrics requires a deep analysis of the spectral properties of adapted transfer operators to deal with the presence of critical points in the Julia sets and the lack of uniform hyperbolicity.

2.1. **Statement of results.** Denote by Poly_D^{cm} (resp. Rat_D^{cm}) the space of critically marked degree $D \geq 2$ polynomials (resp. rational maps). An algebraic subfamily Λ of Poly_D^{cm} (resp. Rat_D^{cm}) is a *Misiurewicz family* if it is the open subset of a family given by a finite number of critical relations of the form

$$f^{n_i}(c_i(\lambda)) = f^{n_i+m_i}(c_i(\lambda)) = r_i(\lambda),$$

where $c_i(\lambda)$ is the i-th marked critical point of f_λ, n_i, m_i are positive integers, and each $r_i(\lambda)$ is a repelling periodic point for every $f_\lambda \in \Lambda$. The dynamics of a Misiurewicz polynomial $f_\lambda \in \Lambda$ on its Julia set is not uniformly hyperbolic, due to the presence of critical points. We say that a stable component $\Omega \Subset \Lambda$ is Λ-*hyperbolic* if for every $\lambda \in \Omega$, every critical point $c_j(\lambda)$ such that c_j is active on Λ is contained in the basin of some attracting cycle for f_λ.

As the first step of our construction of the 2-form $\langle \cdot, \cdot \rangle_G$, we show that the Hausdorff dimension function, i.e., the map sending $\lambda \in \Omega$ to the Hausdorff dimension of the Julia set of f_λ, is real-analytic.

Theorem 2.1. *Let Ω be a Λ-hyperbolic component of a Misiurewicz subfamily Λ of rat_D^{cm}. The Hausdorff dimension function $\delta \colon \Omega \to (0,2)$ sending λ to the Hausdorff dimension of the Julia set of f_λ is real-analytic.*

We then construct the Weil-Petersson metric – a positive semi-definite symmetric bilinear form $\langle \cdot, \cdot \rangle_G$ – on each tangent space $T_\lambda \Omega$. The construction follows the general framework of [HN23; BH24b].

By construction, the 2-form $\langle \cdot, \cdot \rangle_G$ may not be non-degenerate; namely, there may exist a non-zero tangent vector $\vec{v} \in T_\lambda \Omega$ such that $\langle \vec{v}, \vec{v} \rangle_G = 0$. For example, in [HLP24; HLP25], Lee, Park and the second author studied the degeneracy loci of the Weil-Petersson metric on spaces of quasi-Blaschke products. On the other hand, in [HN23], Nie and the second author gave a sharp condition under which the Weil-Petersson metric is non-degenerate on (uniformly) hyperbolic components of rational maps. This result deeply exploited the uniform hyperbolicity and in particular a result by Oh-Winter [OW17] on the distribution of the multiplier of the periodic points. While such tools are not available in our setting, we can still prove that $\langle \cdot, \cdot \rangle_G$ defines a path-metric on Ω.

Theorem 2.2. *Let Λ be a Misiurewicz subfamily of poly_D^{cm} and $\Omega \Subset \Lambda$ a bounded Λ-hyperbolic component. Then the function $d_G : \Omega \times \Omega \to \mathbb{R}$ given by*

$$d_G(x,y) := \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_G dt$$

is a distance function. Here the infimum is taken over all the C^1-paths γ connecting x to y in Ω.

The above theorem parallels the result we obtained in [BH24b] for the case of parabolic families of polynomials. We note that the analysis in [BH24b] does not apply to the case of Misiurewicz maps. More specifically, for parabolic maps, the geometric potential is Hölder continuous, hence we could apply the machinery of [BD23; BD24] to the transfer orerator associated to such weights and their perturbations. For Misiurewicz maps, we instead consider suitable transfer operators acting on the Banach space of Hölder continuous functions on an enlarged tower space, where the dynamics becomes expanding. See also [MS03].

3. Manhattan curves in complex dynamics and asymptotic correlation of multiplier spectra

An alternative way of constructing the pressure metric is to use the so-called *Manhattan curves.* The Manhattan curve for a pair of hyperbolic surfaces (possibly with cusps) has been extensively studied as a way to understand geodesics on surfaces, the thermodynamic formalism of the geodesic flows and comparison of hyperbolic metrics.

The second paper completed during our stay at the Institute concerns Manhattan curves in holomorphic dynamics. In particular, we define and study the Manhattan curve for a pair of hyperbolic rations maps on $\mathbb{CP}^1$, and more generally of holomorphic endomorphisms of $\mathbb{CP}^k$. We prove several counting results for the multiplier spectrum and show that the Manhattan curve for two holomorphic endomorphisms is related to the correlation number of their multiplier spectra.

3.1. Statement of results.

Given a convex cocompact Fuchsian group Γ, the classical *Poincaré series* of Γ is defined as

$$P_\Gamma(s,x) := \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)}$$

for every $s \in \mathbb{R}$ and $x \in \mathbb{H}^2$, where $d(\cdot,\cdot)$ is the hyperbolic distance on $\mathbb{H}^2$. It is well-known [Nic89] that the Poincaré series has a critical exponent δ_Γ for which, for all $x \in \mathbb{H}^2$, $P_\Gamma(s,x)$ converges for $s > \delta_\Gamma$, and diverges for $s < \delta_\Gamma$. The critical exponent is also the Hausdorff dimension of the limit set of Γ as well as the topological entropy of the geodesic flow on the unit tangent bundle of S; see for instance [Sul79; Nic89].

If ρ_1 and ρ_2 are two discrete faithful representations of $\pi_1 S$ into $\mathrm{PSL}(2,\mathbb{R})$, for $(a,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \setminus \{(0,0)\}$, the *weighted Poincaré series for ρ_1 and ρ_2* is defined as

$$P^{a,b}_{\rho_1,\rho_2}(s,x_1,x_2) := \sum_{g\in\pi_1 S} e^{-s(a\cdot d(x_1,\rho_1(g)x_1)+b\cdot d(x_2,\rho_2(g)x_2))}$$

for every $s \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{H}^2$. One can then denote by $\delta^{a,b}_{\rho_1,\rho_2}$ the critical exponent for $P^{a,b}_{\rho_1,\rho_2}$; namely, for every $x_1, x_2 \in \mathbb{H}^2$, $P^{a,b}_{\rho_1,\rho_2}(s,x_1,x_2)$ converges if $s > \delta^{a,b}_{\rho_1,\rho_2}$ and diverges if $s < \delta^{a,b}_{\rho_1,\rho_2}$.

The *Manhattan curve* $\mathcal{C} = \mathcal{C}(\rho_1,\rho_2)$ is defined as

$$\{(a,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \setminus \{(0,0)\} : \delta^{a,b}_{\rho_1,\rho_2} = 1\}.$$

In [Bur93], using the Patterson-Sullivan theory, Burger proved that if $\rho_1, \rho_2 \colon \pi_1 S \to \mathrm{PSL}(2,\mathbb{R})$ are convex cocompact representations, then the Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ is C^1. This result was strengthened by Sharp [Sha98], who used thermodynamic formalism to prove that $\mathcal{C}(\rho_1,\rho_2)$ is in fact real-analytic. Moreover, Manhattan curves have been studied in the context of hyperbolic surfaces with cusps or punctured surfaces, cusped Hitchin representations and left-invariant hyperbolic metrics on a hyperbolic group.

Given a hyperbolic surface $\mathbb{H}^2/\rho_1(\pi_1 S)$ and a conjugacy class $[\gamma]$ in $\pi_1 S$, we denote by $l_1([\gamma])$ the hyperbolic length of the unique closed geodesic in the conjugacy class $[\gamma]$ on the surface $\mathbb{H}^2/\rho_1(\pi_1 S)$. Pollicott-Sharp [PS98] proved the following asymptotic growth rate for the length spectrum of primitive closed geodesics (i.e., geodesics which are not multiples of another geodesic) on $\mathbb{H}^2/\rho_1(\pi_1 S)$: there exists $0 < c < h$ (where h is the entropy of the geodesic flow of $\mathbb{H}^2/\rho_1(\pi_1 S)$) such that

$$\mathrm{Card}\{[\gamma] : l_1([\gamma]) \leq T\} = Li(e^{hT}) + O(e^{cT}), \quad \text{where } Li(y) = \int_2^y \frac{1}{\log u}du. \tag{1}$$

The Manhattan curve $\mathcal{C}(\rho_1,\rho_2)$ is related to the *correlation of the length spectra* on the hyperbolic surfaces $\mathbb{H}^2/\rho_1(\pi_1 S)$ and $\mathbb{H}^2/\rho_2(\pi_1 S)$. If ρ_1 and ρ_2 are not conjugate in $\mathrm{PSL}(2,\mathbb{R})$, Schwartz–Sharp [SS93] proved that there exist two constants $C = C(\epsilon) > 0$ and $\alpha \in (0, 1)$ independent of ϵ such that, for all $\epsilon > 0$, we have

$$\text{Card}\{[\gamma] : l_1([\gamma]), l_2([\gamma]) \in (T, T+\epsilon)\} \sim C\frac{e^{\alpha T}}{T^{3/2}} \quad \text{as} \quad T \to \infty. \tag{2}$$

The number α is called the *correlation number* of ρ_1 and ρ_2. Sharp [Sha98] showed that $\alpha = a + b$ where (a, b) is the point on the curve $\mathcal{C}(\rho_1, \rho_2)$ for which the tangent line has slope -1.

In the context of holomorphic dynamics, McMullen [McM00] introduced a *Poincaré series* $P_f(s, x)$ for every rational map f on the Riemann sphere $\mathbb{P}^1 = \mathbb{CP}^1$ of degree $d \geq 2$ by

$$P_f(s, x) := \sum_{n=1}^{\infty} \sum_{f^n(y)=x} |(f^n)'(y)|^{-s}.$$

The *critical exponent* δ_f is defined as the supremum of those $s \geq 0$ such that $P_f(s, x) = \infty$ for all $x \in \mathbb{P}^1$. McMullen proved that if f is a geometrically finite rational map (i.e., the critical points in the Julia sets are preperiodic), then δ_f equals the Hausdorff dimension $\text{H.dim}(J(f))$ of the Julia set $J(f)$ and the Poincaré series $P_f(s, x)$ is divergent when $s = \delta_f$ for all $x \in \mathbb{P}^1$; see [McM00, Theorem 1.2].

In our second paper, we define the Manhattan curve for a pair of rational maps and more generally a pair of holomorphic endomorphisms of $\mathbb{P}^k = \mathbb{CP}^k$ in the same hyperbolic component and study its regularity properties. In particular, we show that it is real-analytic. For $k \geq 2$, a central object in this study is the *volume dimension* of hyperbolic Julia sets, which we introduced in our earlier paper [BH24a] as an appropriate replacement of the Hausdorff dimension in higher dimensional expanding holomorphic dynamical systems.

We also prove several counting results for the multiplier spectrum in the spirit of (1) and (2) and in particular we show that, under suitable assumptions, the Manhattan curve for two holomorphic endomorphisms is related to the correlation number of their multiplier spectra.

References

[BD23] Fabrizio Bianchi and Tien-Cuong Dinh. "Equilibrium states of endomorphisms of $\mathbb{P}^k$ I: existence and properties". In: *J. Math. Pures Appl. (9)* 172 (2023), pp. 164–201.

[BD24] Fabrizio Bianchi and Tien-Cuong Dinh. "Equilibrium states of endomorphisms of $\mathbb{P}^k$: spectral gap and limit theorems". In: *Geom. Funct. Anal.* 34 (2024), pp. 1006–1051.

[BH24a] Fabrizio Bianchi and Yan Mary He. "A Mañé-Manning formula for expanding measures for endomorphisms of $\mathbb{P}^k$". In: *Trans. Amer. Math. Soc.* 377.11 (2024), pp. 8179–8219.

[BH24b] Fabrizio Bianchi and Yan Mary He. "Pressure path metrics on parabolic families of polynomials". In: (2024). arXiv:2409.10462.

[Bur93] Marc Burger. "Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2". In: *Internat. Math. Res. Notices* 7 (1993), pp. 217–225.

[HLP24] Yan Mary He, Homin Lee, and Insung Park. "Pressure metrics in geometry and dynamics". In: (2024). arXiv:2407.18441.

[HLP25] Yan Mary He, Homin Lee, and Insung Park. "Complex analytic theory on the space of Blaschke products: Simultaneous uniformization and the pressure metric". In: (2025). arXiv:2507.17077.

[HN23] Yan Mary He and Hongming Nie. "A Riemannian metric on hyperbolic components". In: *Math. Res. Lett.* 30.3 (2023), pp. 733–764.

[Ivr14] Oleg Ivrii. *The geometry of the Weil-Petersson metric in complex dynamics.* Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 2014, p. 75.

[McM00] Curtis T. McMullen. "Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps". In: *Comment. Math. Helv.* 75.4 (2000), pp. 535–593.

[McM08] Curtis McMullen. "Thermodynamics, dimension and the Weil-Petersson metric". In: *Invent. Math.* 173.2 (2008), pp. 365–425.

[MS03] N. Makarov and S. Smirnov. “On thermodynamics of rational maps. II. Non-recurrent maps”. In: *J. London Math. Soc. (2)* 67.2 (2003), pp. 417–432.

[Nic89] Peter J. Nicholls. *The ergodic theory of discrete groups.* Vol. 143. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1989, pp. xii+221.

[OW17] Hee Oh and Dale Winter. “Prime number theorems and holonomies for hyperbolic rational maps”. In: *Invent. Math.* 208.2 (2017), pp. 401–440.

[PS98] Mark Pollicott and Richard Sharp. “Exponential error terms for growth functions on negatively curved surfaces”. In: *Amer. J. Math.* 120.5 (1998), pp. 1019–1042.

[Sha98] Richard Sharp. “The Manhattan curve and the correlation of length spectra on hyperbolic surfaces”. In: *Math. Z.* 228.4 (1998), pp. 745–750.

[SS93] Richard Schwartz and Richard Sharp. “The correlation of length spectra of two hyperbolic surfaces”. In: *Comm. Math. Phys.* 153.2 (1993), pp. 423–430.

[Sul79] Dennis Sullivan. “The density at infinity of a discrete group of hyperbolic motions”. In: *Inst. Hautes Études Sci. Publ. Math.* 50 (1979), pp. 171–202.