1. The Erdős–Gallai theorem states that a sequence $(d_1, \dots, d_n)$ of non-negative integers is the degree sequence of a simple graph if and only if the total degree $\sum_{i=1}^n d_i$ is even and for every subset $S \subseteq [n]$, we have
$$\sum_{i \in S} d_i \leq |S|(|S|-1) + \sum_{j \notin S} \min(d_j, |S|).$$
Convince yourself that this implies that d-regular n-vertex graphs exist whenever dn is even and $d \leq n-1$.

2. Show that there are exactly $\frac{n!}{(n/2)!\,2^{n/2}}$ (labelled) perfect matchings on $[n]$, provided n is even.

3. Fix $d \geq 1$ and n such that dn is even. Recall the configuration model from the lecture, that is, let $C_1, \dots, C_n$ be pairwise disjoint sets of size d, and let P be a uniform random perfect matching on $\bigcup_i C_i$.
   (a) Given distinct points $x, y \in \bigcup_{i \in [n]} C_i$, argue carefully why $\Pr(xy \in P) = \frac{1}{nd-1}$.
   (b) For an integer $k \geq 1$, let $x_1, \dots, x_k, y_1, \dots, y_k$ be distinct points in $\bigcup_i C_i$. Show that
$$\Pr(x_i y_i \in P \text{ for all } i) = \prod_{i=0}^{k-1} \frac{1}{nd-2i-1} = \frac{1}{(nd)^k}\left(1 + O\left(\frac{k}{nd}\right)\right).$$

4. For a random pairing P, and $i \geq 1$, let X_i be the number of of cycles of length i in the multigraph $G(P)$, and let $\lambda_i = \frac{(d-1)^i}{2i}$.
   (a) Verify the condition for the method of moments (Lemma 3.4) for $k = 2$. That is, prove that for all fixed non-negative integers r_1 and r_2 we have
$$\mathbb{E}\left([X_1]_{r_1}[X_2]_{r_2}\right) \to \lambda_1^{r_1}\lambda_2^{r_2}$$
as $n \to \infty$.
   (b) Prove Lemma 3.5 from the notes by using the method of moments, that is, generalise your proof from (a) to fixed $k \geq 3$.

5. How do we have to adapt the configuration model to count $|\mathcal{G}(\mathbf{d})|$ for irregular sequences $\mathbf{d}$?
   (a) How does the formula in (1) in the lecture notes change?
   (b) Let X_1 and X_2 count the number of loop pairs and of double-edge pairs, respectively, in this adapted model. Calculate $\mathbb{E}(X_1)$ and $\mathbb{E}(X_2)$. What do you conjecture Pr(Simple) is asymptotically equal to now, assuming $\Delta = \Delta(\mathbf{d}) = O(1)$?
   (c) Prove your conjecture.