1. Rationality,Algebraicity, and Holonomicity

We begin with an easy lemma.

Lemma 1.1. Let $f(x) = \sum a_n x^n \in \mathbf{Z}[\![x]\!]$. Suppose that:

(1) $f(x)$ is holomorphic in a disc of radius $R > 1$.
(2) $a_n \in \mathbf{Z}$.

Then $f(x)$ is a polynomial.

Proof. We have the following estimates on a_n.

(1) A Cauchy upper bound on a_n: By Cauchy's integral formula,

$$a_n = \frac{1}{2\pi i} \oint \frac{f(x)dx}{x^{n+1}}.$$

Take the contour to be a disc of radius η with $1 < \eta < R$ for some fixed η. The function $f(x)$ is uniformly bounded on that disc, say by a constant C. Hence

$$|a_n| \leq \frac{1}{2\pi} \oint_{|x|=\eta} \left| \frac{f(x)}{x^{n+1}} \right| = \frac{2\pi\eta}{2\pi} \frac{C}{\eta^{n+1}} = \frac{C}{\eta^n} = o(1).$$

(2) A Liouville lower bound on a_n: If $f(x)$ is not a polynomial, then $a_n \neq 0$ for infinitely many integers n. For those integers n, we have the bound

$$|a_n| \geq 1.$$

We call this Liouville because in Liouville's estimate on $|\alpha - p/q|$ for a root α of. a degree n polynomial, the key input is that $|f(p/q)q^n| \geq 1$ if it is non-zero.

These bounds are in conflict, hence $f(x)$ must be a polynomial. □

How can one improve on these theorems?

(1) I would like to sketch improvements by Borel, Dwork, Polya, Betrandias,
(2) Vesselin will talk about André's results — introduce our improvement, and explain the new ingredients missing from André.

1.1. Borel.

In the trivial lemma above, let's drop the hypothesis that $f(x)$ is analytic and weaken it to being meromorphic instead.

Theorem 1.2 (Borel)**.** *Let $f(x) = \sum a_n x^n \in \mathbf{Z}[\![x]\!]$. Suppose that:*

(1) $f(x)$ is meromorphic in a disc of radius $R > 1$.
(2) $a_n \in \mathbf{Z}$.

Then $f(x)$ is a rational function.

Here's an incorrect proof. By compactness, $f(x)$ has only finitely many poles in a closed disc of radius η for $1 < \eta < R$. Let $q(x)$ be a polynomial which vanishes at all these points to the correct order — now apply the trivial lemma to $f(x)q(x)$ to deduce that $f(x)q(x) = p(x)$. The problem is that this only works if we can take $q(x) \in \mathbf{Z}[x]$, and this only works if the poles are algebraic numbers. But this doesn't have to be true a priori (but it does a posteriori).

The proof idea (all the proof ideas) are to combine an upper and lower bound. For rationality proofs, one strategy is to find a criteria strictly in terms of integers for when a function $f(x)$ is a rational function. We know that coefficients of rational functions satisfy linear recursion relations (maybe with transcendental coefficients). It turns out that the key idea is to look at the determinants

$$\begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+n} \\ a_{k+1} & a_{k+2} & \dots & a_{k+1+n} \\ & \ddots & & \\ a_{k+n} & a_{k+n+1} & \dots & a_{k+2n-1} \end{vmatrix}.$$

The vanishing of these determinants detects whether $f(x)$ is rational or not. For example, the coefficients a_n form an arithmetic progression if and only if the 3×3 determinants vanish; remember back to when you teach linear algebra and write down a random matrix to invert:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

and then find to your horror that it has determinant zero!

Let me explain the argument the case where there is a single pole at some transcendental point λ where $|\lambda| < 1$. (or $\lambda \le 1$ as well. The case $|\lambda| > 1$ already follows by the trivial lemma). We have bounds

$$|a_n| = O(1/\lambda^n)$$

which could be big. But if $\sum b_n = f(x)(x - \lambda) = \sum x^n(\lambda a_n - \lambda a_{n-1})$, then

$$|b_n| = O(1/\eta^n).$$

is small. And now

$$\begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+n} \\ a_{k+1} & a_{k+2} & \dots & a_{k+1+n} \\ & \ddots & & \\ a_{k+n} & a_{k+n+1} & \dots & a_{k+2n-1} \end{vmatrix} = \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+n} \\ a_{k+1} - \lambda a_k & a_{k+2} - \lambda a_{k+1} & \dots & a_{k+1+n} - \lambda a_{k+n} \\ & \ddots & & \\ a_{k+n} - \lambda a_{k+n-1} & a_{k+n+1} - \lambda a_{k+n} & \dots & a_{k+2n-1} - \lambda a_{k+2n-2} \end{vmatrix}$$

$$= \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+n} \\ b_k & b_{k+1} & \dots & b_{k+n} \\ & \ddots & & \\ b_{k+n-1} & b_{k+n} & \dots & b_{k+2n-2} \end{vmatrix}.$$

If n is big enough with respect to η and λ, this is both very small and an integer, hence zero.

1.2. **Dwork.** The above argument work exclusively at the complex place, but there is a p-adic version as well, and versions which mix primes. Note however that any rational function in $\mathbf{Q}[\![x]\!]$ — indeed any algebraic function, has coefficients in $\mathbf{Z}[1/S]$ for some integer S.

1.3. **Polya–Bertrandias.** Here we weaken the hypothesis that $f(x)$ extends to a meromorphic function for $R > 1$, but we still assume *some* convergence outside the disc. To give a completely concrete example, suppose that $f(x) \in \mathbf{Z}[\![x]\!]$ extended to a holomorphic function on $\mathbf{C} \setminus [1, \infty)$. We can't use Cauchy on any disc of radius $\eta > 1$, so where do we get a good upper bound from?

Lemma 1.3. The only such functions have the form $P(x)/(1-x)^N$ for some integer N.

Proof. Consider

$$\oint \left(\frac{1}{x^2} - \frac{1}{x}\right)^n \frac{f(x)}{x} dx, \quad \oint \left(\frac{1}{x^2} - \frac{1}{x}\right)^n f(x) dx.$$

These integrals vanish for n large enough. The reason is that if we pick a contour close enough to the boundary (a fixed contour), we see that the function $f(x)$ is bounded, but the function $(1/x^2 - 1/x)$ is bounded above by $(1/4 + \varepsilon)$, and so this is very small. But it is also an integer, and hence it must be zero for n large enough.

After replacing $f(x)$ by $f(x)(1/x - 1/x^2)^N \in \mathbf{Z}[\![x]\!](1/x)$ for large enough N, the integrals above vanish for all $n \in \mathbf{N}$.

We can write

$$\oint \frac{f(x)}{x^{2n+1}} = \oint \left(\frac{1}{x^2} - \frac{1}{x}\right)^n \frac{f(x)}{x} dx + \sum_{i<2n+1} c_{2n+1,i} \oint \frac{f(x)}{x^i},$$

$$\oint \frac{f(x)}{x^{2n}} = \oint \left(\frac{1}{x^2} - \frac{1}{x}\right)^n f(x) dx + \sum_{i<2n} c_{2n_i} \oint \frac{f(x)}{x^i},$$

and hence by induction deduce that $f(x) \in \mathbf{Z}[1/x]$, and thus

$$f(x) = \frac{P(x)}{(1-x)^N}$$

for some polynomial $P(x)$. □

This proof works as long as $1/x^2 - 1/x$ has absolute value less than one near the boundary of the region, or equivalently that $x - x^2$ has absolute value less than one near the interval $[0, 1]$ after replacing x by $1/x$. This means that exactly the same argument works for $\mathbf{C} \setminus [\alpha, 0]$ where

$$\alpha > \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2} = 0.618\ldots$$

because $|x - x^2| < 1$ on $[0, 1/\alpha]$ for $1/\alpha < \phi$. This is optimal, because

$$\frac{1}{1 - x - x^2} \in \mathbf{Z}[\![x]\!]$$

But you can replace $x - x^2$ by

$$(x - x^2)(x - x^2 + 1)$$

and get a larger interval, and hence deduce rationality for even smaller α. So it's all about finding monic polynomials $f(x)$ which are less than 1 on the interval $[0, 1/\alpha]$, which relates (almost by definition) to the transfinite diameter of this interval $[0, 1/\alpha]$. Since $\mathbf{C} \setminus [\alpha, \infty)$

is simply connected, this can also be directly related to the conformal radius of the region. In particular, make the substitution

$$z \mapsto \frac{-4z}{(1-z)^2}$$

which maps $D(0,1)$ to $\mathbf{C} \setminus [1, \infty)$, and realizes this region with conformal radius 4. If you replace $\mathbf{C} \setminus [1,\infty)$ by $\mathbf{C} \setminus [\alpha, \infty)$ then the conformal radius becomes 4α. So this works for $\alpha > 1/4$, and more generally:

Theorem 1.4 (Polya–Bertrandias). *Suppose that $f(x)$ extends as a meromorphic function to a region of conformal radius $R > 1$. Then $f(x)$ is rational.*

(Vesselin tells me that this is Polya's theorem and Bertrandias' contribution was the p-adic extension which is not included here.)

1.4. **A small cheat.** Note that in our example above we used that the polynomial $x^2 - x$ had coefficients in $\mathbf{Z}$. Having small conformal radius only guarantees the existence of a monic polynomial with arbitrary coefficients which is small. Yet we really used integrality in our arguments to deduce that a small integer had to be zero. If we replace it by $P(x)$, we deduce that

$$\oint P(x)^n f(x) x^i dx$$

is very small for $i = 0, \ldots, \deg(P) - 1$ and for all large enough n. But it doesn't have to be zero because it's not an integer! However, we can once again study the Hankel determinants rather than the coefficients themselves, and deduce that $f(x)$ is rational. (In effect, the version of the argument above when $x - x^2$ happens to have integral coefficients is a little like the case of Borel's theorem where the poles are at algebraic points.)

The example above is optimal in that

$$\sqrt{1-4x} \in \mathbf{Z}[[x]].$$

is not rational.

In all honestly, I'm doubt I will get much further than this point, so the rest is (even more) sloppy than what came before

1.5. **Algebraicity.** Looking at the example $\sqrt{1-4x}$, it has a stronger property than being defined on $\mathbf{C} \setminus [1/4, \infty)$. Namely, it has an analytic continuation to the universal cover of $\mathbf{C} \setminus \{0, 1/4\} = \mathbf{P}^1 - \{0, \alpha, \infty\}$ for $\alpha = 1/4$. Once more, let us consider the special case of mutivalued functions on $\mathbf{P}^1 - \{0, \alpha, \infty\}$ with coefficients in $\mathbf{Z}[\![x]\!]$. Algebraicity implies some $\mathbf{Q}[x]$-linear relationship between the powers $f(x)^n \in \mathbf{Z}[\![x]\!]$.

1.6. **The stragegy.** Form a $\mathbf{Z}[x]$ linear combination of these powers:

$$F(x) = \sum a_i(x) f^i(x)$$

such that:

(1) The polynomials $a_i(x)$ have moderate degree
(2) The sum $F(x)$ vanishes to high order at $x = 0$.

(3) Make the coefficients of $a_i(x)$ not too big.

If we can do this, then with $F(x) = \sum a_n x^n$ we can play the game with bounding a_n for the smallest non-zero coefficient. It's non-zero assuming that $f(x)$ is not algebraic, but then obtain a Cauchy bound.

There is tension between these conditions. For the first two, bounding the degree and imposing vanishing is simply a system of linear equations. If we have enough equations (high enough degree) we get a solution. However, those solutions might be very large. So how do we control these solutions and hence the coefficients of $a_i(x)$? The key point is to have many more variables than equations so the solution space is big — for example, take twice as many variables as equations. We still get vanishing to high degree, but we have much more flexibility in looking for a small solution. Indeed, the classic Siegel's lemma is exactly about finding small solutions to such underdetermined linear equations.

1.7. **André, Limits of this theorem.**

Theorem 1.5. *If $f(x)$ is a multivalued function on $\mathbf{P}^1 \setminus \{0, \alpha, \infty\}$ which is holomorrhic at (the original branch of 0, and $\alpha > 1/16$, then $f(x)$ is algebraic.*

Actually, this is best possible. Take

$$F(1/2, 1/2, 1, 16x) = \sum \binom{2n}{n}^2 x^n \in \mathbf{Z}[\![x]\!].$$

is a solution to an ODE in $\mathbf{P}^1 \setminus \{0, 1/16, \infty\}$. But I claim it is not an algebraic function. We saw that

$$F(1/2, 1/2, 1, 16\lambda/16) = F(1/2, 1/2, 1, \lambda) = \left(1 + 2q + 2q^4 + 2q^9 + \ldots\right) = \theta^2(\tau)$$

Now $\theta^2(\tau)$ has weight one and is not an algebraic function of λ which has weight zero.

In this particular example, the monodromy around 0 (of the other solution!) is unipotent. The solutions are $\{1, \tau\}$, and around a loop we should get $\tau+2$, since that doesn't change $q = e^{\pi i \tau}$. The monodromy is actually $\Gamma(2)$ since it is the Picard–Fuchs equation.