ARITHMETIC OF AUTOMORPHIC FORMS AND L-FUNCTIONS

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We are grateful to the School of Mathematics, Institute for Advanced Study, for the opportunity for the six of us to come together for four weeks in the month of July, 2023. During this time, various subsets of our team worked on the following general themes:

(1) Congruences for the special values of Rankin–Selberg L-functions.
(2) Nonvanishing and arithmetic origin of cuspidal cohomology.
(3) The Langlands–Shahidi L-functions.

The overarching framework unifying these three themes is an interpretation of the Langlands–Shahidi theory of L-functions in terms of Eisenstein cohomology. The entire investigation being carried out with a view towards a general arithmetic theory of automorphic L-functions. In the following three sections we present the results of our investigations.

1. Congruences for the special values of L-functions

It is a well-known principle in number theory that a congruence between objects translates to a congruence between the special values of L-functions attached to these objects. We investigated this principle for Rankin–Selberg L-functions attached to pairs of holomorphic cuspforms. Suppose we have two primitive holomorphic cusp forms $f, f' \in S_k(\Gamma_1(N))$ of weight k and level N. Suppose $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ and $f'(z) = \sum_{n=1}^{\infty} a'_n e^{2\pi i n z}$ are their Fourier expansions. Let $\mathbb{Q}(f)$ (resp., $\mathbb{Q}(f')$) denote the number field generated by the Fourier coefficients of f (resp., f'), and take F to be any number field containing $\mathbb{Q}(f)$ and $\mathbb{Q}(f')$. Let $\mathfrak{P}$ be a prime ideal of the ring of integers of F. We say f is congruent to f' modulo $\mathfrak{P}$, denoted as $f \equiv f' \pmod{\mathfrak{P}}$, if $a_n \equiv a'_n \pmod{\mathfrak{P}}$ for all $n \geq 1$. Suppose $g \in S_l(\Gamma_1(M))$. Assume that $k > l$. Then the Rankin–Selberg L-functions $L(s, f \times g)$ and $L(s, f' \times g)$ have $\{m \in \mathbb{Z} : l \leq m \leq k-1\}$ as their set of critical points. For any of these critical points, one expects a congruence between the algebraic parts of $L(m, f \times g)$ and $L(m, f' \times g)$. One has to be careful in clarifying the meaning of the algebraic part of an L-value. However, keeping in mind the shape of the rationality results due to Shimura for these L-values, one sees that if we had at least two critical points, then the ratio $L(m, f \times g)/L(m+1, f \times g)$ of successive critical values for the completed L-function lies in the compositum of the number fields $\mathbb{Q}(f)$ and $\mathbb{Q}(g)$. Let us enlarge the number field F above also to contain $\mathbb{Q}(g)$. We expect that:

$$f \equiv f' \pmod{\mathfrak{P}} \implies \frac{L(m, f \times g)}{L(m+1, f \times g)} \equiv \frac{L(m, f' \times g)}{L(m+1, f' \times g)} \pmod{\mathfrak{P}}.$$

There are some obvious variations to consider which is to freeze the larger weight cuspform f and take two congruent cuspforms g, g' of smaller weight; or to consider a congruence between a cuspform and an Eisenstein series such as the celebrated Ramanujan congruence. We have computationally verified that the above implication on congruences holds in the following cases:

(1) $f, f' \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, $\mathfrak{P} = 144169$, $g = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
(2) $f, f' \in S_{30}(\mathrm{SL}_2(\mathbb{Z}))$, $\mathfrak{P} = 51349$, $g = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
(3) $f, f' \in S_{13}(\Gamma_0(3), \chi)$, $\mathfrak{P} = 13$, $g = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
(4) $f \in S_{26}(\mathrm{SL}_2(\mathbb{Z}))$, and $g, g' \in S_{13}(\Gamma_0(3), \chi)$, $\mathfrak{P} = 13$.
(5) $f \in S_{24}(\mathrm{SL}_2(\mathbb{Z}))$, $g = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$, g' a weight 12 Eisenstein series, $\mathfrak{P} = 691$.

The results are being written up and will appear as [6]. We are currently working on a proof of such a congruence for L-functions for $\mathrm{GL}(2) \times \mathrm{GL}(2)/\mathbb{Q}$ using the methods of Eisenstein cohomology as in [5]. The proof will be carried out with a view toward generalizations: replacing $\mathbb{Q}$ by a totally real number field (Hilbert modular forms) and replacing $\mathrm{GL}(2) \times \mathrm{GL}(2)$ by higher groups, such as $\mathrm{GL}(n) \times \mathrm{GL}(m)$ with nm even, and $\mathrm{O}(n, n)$ for an even n. One seemingly unavoidable caveat while using cohomologial methods to prove congruences is that we have to exclude a finite set of primes that support torsion in integral cohomology.

A closely related project is to generalize in a different direction by replacing f and g by Hida families containing them. This is a long term project for which we are still working on the foundational aspects [1].

2. Cuspidal cohomology for GL(n) over a number field

Let F be a number field, and $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}(n))$. Suppose E is a Galois number field that is large enough to contain a copy of F; then E splits G. For a dominant integral weight $\lambda \in X^*(\mathrm{Res}_{F/\mathbb{Q}}(T_n) \times E)$, where T_n is the diagonal torus of $\mathrm{GL}(n)$, let $\mathcal{M}_{\lambda,E}$ be the finite-dimensional absolutely-irreducible representation of $G \times E$ with highest weight λ. For an open-compact subgroup K_f of $G(\mathbb{A}_f)$ let $S^G_{K_f}$ denote the locally symmetric space for G of level K_f, and let $\widetilde{\mathcal{M}}_{\lambda,E}$ be the sheaf of E-vector spaces on $S^G_{K_f}$. An embedding $\iota : E \to \mathbb{C}$, gives the sheaf $\widetilde{\mathcal{M}}_{\iota\lambda,\mathbb{C}}$ of $\mathbb{C}$-vector spaces on $S^G_{K_f}$. A basic object of interest in the arithmetic theory of automorphic forms is the cuspidal cohomology $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\iota\lambda,\mathbb{C}})$. For $F = \mathbb{Q}$ and $n = 2$, this corresponds via the Eichler-Shimura isomorphism to a suitable space of holomorphic cuspidal modular forms.

2.1. In one of the projects, we studied the fundamental problem of nonvanishing of cuspidal cohomology. It is somewhat surprising that even for GL(2) over a general number field, this problem is not fully settled. For general n, if F is totally real or is a CM field, then some partial results are available in the literature. A necessary condition for nonvanishing of $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\iota\lambda,\mathbb{C}})$ is that λ is strongly-pure, a kind of conjugate-self-duality condition on λ which was explicated in [7]. For general n and F, and for a strongly-pure weight λ, the problem of proving nonvanishing of cuspidal cohomology seems to be wide open. In this context, we proved the following theorem [4].

Theorem 2.1. *Assume that the number field F is Galois over $\mathbb{Q}$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}(2))$. Suppose $\lambda \in X^*(\mathrm{Res}_{F/\mathbb{Q}}(T_2) \times E)$ is a strongly-pure weight, then for any $\iota : E \to \mathbb{C}$, there is a deep enough open-compact subgroup K_f such that $H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\iota\lambda,\mathbb{C}}) \neq 0$.*

2.2. For $\mathrm{GL}(n)$ over a totally-real number field F, closely related to cuspidal cohomology is the the notion of strongly-inner cohomology, denoted $H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$, that was proposed in [5] as an arithmetic precursor of cuspidal cohomology. First of all, there is inner-cohomology $H^\bullet_{!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$, which is defined to be the image of compactly supported cohomology in total cohomology. Then strongly-inner cohomology $H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})$ was defined as the subspace of inner-cohomology given by the sum of isotypic subspaces for the action of a Hecke algebra for those Hecke modules that can occur only in cohomology between certain specific degrees: $b^F_n \leq \bullet \leq t^F_n$ called the cuspidal-range for $\mathrm{GL}(n)/F$. A basic result proved in [5] is that for any $\iota : E \to \mathbb{C}$, one has

$$H^\bullet_{!!}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \otimes_{E,\iota} \mathbb{C} \; = \; H^\bullet_{\mathrm{cusp}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{{}^\iota\lambda,\mathbb{C}}).$$

In one of the projects [2], we generalized the above construction and result of [5] to the context of $\mathrm{GL}(n)/F$ for *any* number field F, giving another proof of a well-known result of Clozel [3] that cuspidal cohomology has a rational structure. For $F = \mathbb{Q}$ and $n = 2$, this is the analogue of the classical result that the space of weight k level N cusp forms has a basis of forms with rational Fourier coefficients.

3. Enhanced list of Langlands–Shahidi L-functions

The entire list of Langlands–Shahidi L-functions consists of a total of 58 families of L-functions as enumerated in Appendices A, B, and C of Shahidi's book [9]. One of the long-term goals is to study the arithmetic properties of the special values of these L-functions. For each family, what is available in these appendices, is the Dynkin diagram for an ambient connected reductive group G, the deleted root for a maximal parabollc subgroup P, and the highest-weights of the irreducible subrepresentations for the adjoint action of the dual ${}^LM$ on the Lie algebra ${}^L\mathfrak{n}$ of the unipotent radical for the dual parabolic subgroup ${}^LP$ of the Langlands dual group ${}^LG$. One needs more information for each L-function to study their special values. For the simplest of situations, when G is of type A_n, then experience says that it is best to work with $G = \mathrm{GL}(n)$ rather than with $\mathrm{SL}(n)$ or $\mathrm{PGL}(n)$ to study Rankin–Selberg L-functions. A little more striking is the example of Asai L-functions, where one knows that it is better to take $G = \mathrm{GU}(n,n)$ rather than $\mathrm{U}(n,n)$, to get information about all critical values for twisted Asai L-functions. With such examples in mind, and with a view towards an arithmetic theory of L-functions, for each L-function, we are explicating the best possible ambient group and building a check-list of properties for that context to satisfy so that one may study the arithmetic aspects of that given L-function. This is a long and tedious exercise with results getting written up in [8]. Each case offers its own peculiarities, and with several exotic L-functions showing up; for example, the degree-16 half-spin L-function for $\mathrm{Spin}(10)$ appears by studying E_6; the L-function of non-normal cubic base-change for $\mathrm{GL}(2)$ or the non-normal cubic Asai L-function appear in various quasisplit trialitarian D_4's.

4. A mini-symposium

A one-day internal symposium was held on the 18th of July. Talks were delivered by Raghuram, Bhagwat, Balasubramanyam, and Shahidi. Whereas the first three talks were meant only for our team of collaborators, the talk by Freydoon Shahidi was announced as a special seminar, and was attended by other members and faculty of the IAS, a few faculty and postdocs/students from Princeton University, with some of the attendees joining the talk by Zoom.

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