10 YEARS SINCE THE IAS GALOIS THEORY AND AUTOMORPHIC FORMS SPECIAL YEAR

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I have been asked to give a one page report on how this special year influenced the development of the subject over the next 10 years. It is very hard to trace the development of mathematical ideas, but two things can be said with fair certainty: the subject area has continued to flourish and there were some important developments in the immediate aftermath of the special year which can be clearly linked with the special year. Further out the links are harder to establish, though it would seem surprising if the currents set in motion at the special year haven't continued to be important. The general area has been so fecund it is impossible to survey everything that has happened, so I will simply write a very idiosyncratic account of the important developments which I know best. There is much else I could have written about instead.

Over the last 10 years probably the two most important developments in the automorphy of Galois representations have been the extension of modularity lifting techniques to cases where the basic Taylor-Wiles numerology breaks down and the extension to cases where the image of the residual Galois representation is small. The first of these developments was pioneered by Calegari and Geraghty. Their work began prior to the special year, but both authors were in residence at the special year and Calegari tells me that 'many of the ideas were found during the year' and the first preprint appeared a year later. This line of work has led to two more recent major developments: the proof of the meromorphic continuation and functional equation of the L-functions of all elliptic curves over CM fields (and of the Ramanujan conjecture for the cohomology of Bianchi manifolds) by a group of 10 mathematicians who met at the IAS in 2016; and the proof of the meromorphic continuation and functional equation of curves of genus two over totally real fields by Boxer, Calegari, Gee and Pilloni. The study of modularity lifting in the case of small residual image was started by Skinner and Wiles before the special year, but major advances were made by Thorne, who attended the special year as a graduate student, and later by Pan. At the special year Thorne met Clozel and they began working together on the automorphy (not potential automorphy) of the symmetric power L-function of an elliptic curve over the rational numbers. In subsequent years they proved the automorphy of the fifth to eighth symmetric power L-functions. Very recently Newton and Thorne added new ideas and were spectacularly able to complete the proof of automorphy for all symmetric powers.

At the workshop during the special year Peter Scholze spoke about his newly introduced perfectoid spaces and serendipitously Jared Weinstein, in the next lecture, spoke about his work on explicit integral models for the higher covers of rank 2 Lubin-Tate space. They realized that perfectoid spaces were the correct framework to think about Rapoport-Zink spaces (eg Lubin-Tate space) at infinite level. This led to their celebrated paper on 'moduli of p-divisible groups' in which inter alia they provide a combinatorial description of Rapoport-Zink spaces at infinite level. Also during the special year Harris, Lan, Taylor and Thorne constructed l-adic representations for regular algebraic cuspidal automorphic representations over CM fields (with no self-duality assumption). Scholze, inspired by this, went on to use his theory of perfectoid spaces, this time applied to infinite level Shimura varieties, to construct Galois representations for all eigenclasses in the Betti cohomology of locally symmetric spaces for $GL(n)$ over any CM field. The key point being that amazingly his work included torsion classes. This was a vital ingredient in the work on L-functions for elliptic curves over CM fields alluded to above. Shortly afterwards Boxer gave (unpublished) a different construction of these torsion Galois representations using a different method. Caraiani and Scholze took this perfectoid analysis of Shimura varieties at infinite level even further and used it to give a third construction of these Galois representations, one that had the crucial advantage of enabling them to keep track of local-global compatibility at l. This was another vital ingredient in the work on L-functions of elliptic curves over CM fields.