If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet.
– Niels Bohr

Problem Set, Day One

PCMI USS, Summer 2023

1. **(The Z operator)**
   (a) The operator that takes $|0\rangle$ to $|0\rangle$ and $|1\rangle$ to $-|1\rangle$ is called Z. This operator occurs very frequently in quantum algorithms.
   (b) Write the matrix of Z in the computational basis.
   (c) Is Z unitary?
   (d) Show that Z and H are *unitarily equivalent*. This means that there exists a unitary matrix U such that $Z = U^\dagger H U$. Find such a U.
   (e) Your friend Jeff Hashfield has recently become very excited about the Z operator. Jeff enters and announces,"I came up with a really cool proof that Z is actually the identity operator. Do you want to see it?" Without waiting for an answer, Jeff continues, "Z fixes $|0\rangle$ by definition. Also, Z fixes $|1\rangle$, since $Z|1 = -|1\rangle$. And $-|1\rangle$ is the same as $|1\rangle$ up to a global phase. Since Z fixes a basis and is linear, Z must be the identity map."

   Do you agree with Jeff? Is he essentially right, or can applying Z have a meaningful effect on a quantum system. Explain.

2. Consider these two 2-qubit states

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)$$
$$|\psi_2\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle).$$

   (a) Suppose $|\psi_1\rangle$ is measured in the computational basis. What are the possible outcomes, and what are their probabilities? For each outcome, to what state $|\psi'\rangle$ does $|\psi_1\rangle$ collapse?
   (b) Repeat for $|\psi_2\rangle$.
   (c) Now say you measure only the first qubit of $|\psi_1\rangle$. Aagin describe the possible outcomes, probabilities, and collapsed states.
   (d) Repeat (c) for measurement of first qubit of $|\psi_2\rangle$.
   (e) What observations can you make about your answers to these questions? Why do you think this happened?

3. You are gifted a qubit $|\psi\rangle$ and told that there is a 50% chance that $|\psi\rangle = |0\rangle$ and a 50% chance that $|\psi\rangle = |+\rangle$.

   (a) If you measure $|\psi\rangle$ in the computational basis and the result of the measurement is 0, what is the probability that the state you were gifted was $|0\rangle$?

   (b) If you measure $|\psi\rangle$ in the computational basis and the result of the measurement is 1, what is the probability that the state you were gifted was $|0\rangle$?

   (c) Suppose that you are given $|\psi\rangle$, and you measure in the computational basis. Based on the result of the measurement, you are now supposed to make your best guess as to whether the state you were $|0\rangle$ state or a $|+\rangle$ state. With what probability can you succeed in this task?

   (d) Suppose that when you recieve $|\psi\rangle$, you decide to apply a Hadamard transformation before measuring in the computational basis. (Or equivalently, you decide to measure in the $\{|+\rangle, |-\rangle\}$ basis.) Again you want to determine whether the original state was $|0\rangle$ or $|+\rangle$. With what probability can you succeed?

   (e) Can you find a better basis to measure in? That is, is there a 2×2 unitary U such that applying U and then measuring in the computational basis allows you succeed with higher probability than you found in parts (c) and (d)?

4. Read through the "Complex Inner Product Spaces" handout and work the exercises. This is important background material. Some of this we'll go over in Tuesday's lecture.