Physics 539 - Problem Set 1 - Due September 21

(1) Recall that in class, in two-dimensional Minkowski space $M$ with coordinates $(t, x)$ and metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$, we considered the sequence of curves $\gamma_n$, $n = 1, 2, 3, \cdots$, defined by

$$x = \sin \pi n t$$

If we consider only the portion of these curves with $0 \leq t \leq 1$, we can regard them as non-causal curves from $q = (0, 0)$ to $p = (1, 0)$. As such, they have no limit.

On the other hand, let us consider the same curves over the larger range $0 \leq t < \infty$. As such, we can consider them as inextendible curves in $M$ that all start at the point $q$. We pick a complete Euclidean metric on $M$ (such as $\mathrm{d}s_E^2 = \mathrm{d}t^2 + \mathrm{d}x^2$) and parametrize the $\gamma_n$ by Euclidean arclength. In class we proved that a sequence of inextendible curves from a fixed initial point (whether causal or not), parametrized in this way by arclength in a complete Euclidean metric, always has a convergent subsequence.

In this exercise, you will resolve the tension between these different statements.

(a) Viewing the $\gamma_n$ as inextendible curves with $0 \leq t < \infty$, find a convergent subsequence and describe what it converges to.

(b) Is your answer in (a) a curve from $q$ to $p$ (which might contradict the statement that the $\gamma_n$, viewed as curves from $q$ to $p$, do not have a limit)?

(2) Recall that $\mathcal{C}_q^p$ is the space of causal curves from a point $q$ to a point $p$ in its future. Suppose that $M$ is globally hyperbolic with Cauchy hypersurface $S$. In class, we showed that $\mathcal{C}_q^p$ is compact if $q, p$ are both to the past or both to the future of $S$. Show that this is also true if $q$ is to the past of $S$ and $p$ is to its future.

(3) Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i,j=1}^{d} g_{ij}(t,\vec{x})\mathrm{d}x^i\mathrm{d}x^j.$$

(The significance of this metric will be discussed in class.) View $g_{ij}$ as a $d \times d$ matrix ($d = D - 1$, where $D$ is the spacetime dimension) and write $g^{-1}$ for the inverse matrix, and $\dot{g}$ for $\partial g/\partial t$. Verify that

$$R_{tt} = -\frac{1}{2}\partial_t \mathrm{Tr}\, g^{-1}\dot{g} - \frac{1}{4}\mathrm{Tr}\,(g^{-1}\dot{g})^2.$$

As we will discuss in class, this formula is the key step in deriving Raychaudhuri's equation.