Multisummable functions and o-minimality

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the questions

$\mathcal{R}$ an o-minimal expansion of the real field

$f : (a, \infty) \longrightarrow \mathbb{R}$ a function

Question 1

Is f definable in $\mathcal{R}$?

Examples: Are $\Gamma\restriction_{(a,\infty)}$ or $\zeta\restriction_{(a,\infty)}$ definable in $\mathbb{R}_{\mathrm{an,exp}}$, for some $a > 0$?

Question 2

If f has a holomorphic continuation $\mathbf{f}$ on some complex domain, is such a continuation definable in $\mathcal{R}$?

Examples: The restriction of $\exp$ to any im-bounded domain is definable in $\mathbb{R}_{\mathrm{an,exp}}$. Since Γ and ζ are meromorphic in $\mathbb{C}$, their restrictions to any bounded domain are definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Theorem [Van den Dries, Macintyre and Marker 1990s]: There is a differential ordered field embedding

$$T : \mathcal{H}_{\text{an,exp}} \longrightarrow \mathbb{T}.$$

This embedding has additional properties:

(i) The image $\mathbb{T}_{\text{an,exp}}$ of $\mathcal{H}_{\text{an,exp}}$ under T is truncation closed.

(ii) T **represents asymptotic expansion**, that is, for $f \in \mathcal{H}_{\text{an,exp}}$ and any transmonomial $m \in \text{supp}(Tf)$, we have

$$f - T^{-1}((Tf)_m) = o\left(T^{-1}(m)\right).$$

(iii) Every series in $\mathbb{T}_{\text{an,exp}}$ has finitely generated support.

(iv) If $F \in \mathbb{T}_{\text{an,exp}}$ is a generalized power series, then F converges.

Example 1 [DMM]: $\zeta\restriction_{(a,\infty)}$ is not definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Proof. If it were definable, then so is the function

$$\theta(x) = \zeta(\log x),$$

which is given by the convergent generalized power series $\Theta = \sum_{n=1}^{\infty} x^{-\log n}$. Since T represents asymptotic expansion, it follows that Θ is a truncation of $T\theta$. Since $\mathbb{T}_{\mathrm{an,exp}}$ is truncation closed, we get $\Theta \in \mathbb{T}_{\mathrm{an,exp}}$. But the support of Θ is not finitely generated, which contradicts (iii). □

Example 2 [DMM]: $\Gamma\restriction_{(a,\infty)}$ is not definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. If it were definable, then so is the function

$$e^{\varphi(x)} = \frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}-x} e^x \Gamma(x).$$

From Sterling's Theorem and Borel-Laplace summation, one shows that φ has a *divergent* asymptotic expansion $\Phi \in \mathbb{R}[\![\frac{1}{x}]\!]$. Since T represents asymptotic expansion, it follows that Φ is a truncation of $T\varphi$. Since $\mathbb{T}_{\text{an,exp}}$ is truncation closed, we get $\Phi \in \mathbb{T}_{\text{an,exp}}$. But then the divergence of Φ contradicts (iv). □

$\zeta\restriction_{(1,\infty)}$ is definable

$\mathrm{an}^\omega = (\mathrm{an}_n^\omega)_{n\in\mathbb{N}}$ class of all germs at 0^+ defined by convergent generalized power series with natural support. an_1^ω contains the function θ above

$T : \mathrm{an}_n^\omega \longrightarrow \mathbb{R}[\![X_1^*, \dots, X_n^*]\!]$ the corresponding embedding (inverse of summation)

$\mathbb{R}_{\mathrm{an}^\omega}$ expansion of the real field by all representatives of germs in an^ω. This expands $\mathbb{R}_{\mathrm{an}}$

Theorem [DS 1998]: The structure $(\mathbb{R}_{\mathrm{an}^\omega}, \exp)$ is o-minimal and defines $\zeta\restriction_{(1,\infty)}$.

$\Gamma\restriction_{(0,\infty)}$ is definable

$\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ class of all germs at 0^+ of multisums of *real* power series that are multisummable in the positive real direction. $\mathcal{G}_1$ contains the function φ above:

Fact (early 1900s, see e.g. [Sauzin 2016]): $\varphi(1/x)$ extends to a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$, and φ is a 1-sum of Φ on every closed and bounded subsector of $\mathbb{C} \setminus (-\infty, 0]$.

$T : \mathcal{G}_n \longrightarrow \mathbb{R}[\![X_1, \dots, X_n]\!]$ the corresponding embedding (inverse of Borel-Laplace summation)

$\mathbb{R}_{\mathcal{G}}$ expansion of the real field by all restricted representatives of germs in $\mathcal{G}$. This expands $\mathbb{R}_{\text{an}}$

Theorem [DS 2000]: The structure $(\mathbb{R}_{\mathcal{G}}, \exp)$ is o-minimal and defines $\Gamma\restriction_{(0,\infty)}$.

definability of φ along other directions

Lemma 1: For $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the restriction of φ to $(0, \infty)e^{i\alpha}$ is definable in $\mathbb{R}_{\mathcal{G},\exp}$.

Proof. Set $\varphi_\alpha(z) := \varphi\left(e^{i\alpha} z\right)$. Then φ_α is 1-summable in the positive real direction, and so are $\varphi_\alpha(z) + \overline{\varphi_\alpha(\overline{z})}$ and $i\left(\varphi_\alpha(z) - \overline{\varphi_\alpha(\overline{z})}\right)$. So $\mathrm{re}\,\varphi_\alpha\restriction_{(0,\infty)}$ and $\mathrm{im}\,\varphi_\alpha\restriction_{(0,\infty)}$ are definable in $\mathbb{R}_{\mathcal{G}}$. □

If $|\alpha| \geq \frac{\pi}{2}$, then this proof doesn't work anymore, because $\varphi_\alpha(z) + \overline{\varphi_\alpha(\overline{z})}$ and $\varphi_\alpha(z) - \overline{\varphi_\alpha(\overline{z})}$ are holomorphic only on domains contained in the right half-plane.

definability of φ along other directions

Remarks:

1. The function φ shows that the series $\Phi(X)$ is 1-summable in every direction α satisfying $|\alpha| < \frac{\pi}{2}$.
2. The series $\Phi(X)$ is of the form $\sum_{k \geq 1} a_k X^{2k-1}$, with each $a_k > 0$. So
$$-\Phi(-X) = \Phi(X),$$
and $\Phi(X)$ is also 1-summable in the negative real direction, with corresponding multisum $-\varphi(-z)$. In particular, $\Phi(X)$ is 1-summable in every direction $\alpha \in \left(\frac{\pi}{2}, 3\frac{\pi}{2}\right)$.
3. Since $\Phi(X)$ is divergent, it must have at least one Stokes direction. Hence at least one of i and $-i$ is a Stokes direction; since $\overline{\varphi(\overline{z})} = \varphi(z)$, it follows that they both are.

Question: Are the restrictions of φ to $\pm i(0, \infty)$ definable in $\mathbb{R}_{\mathcal{G},\exp}$?

embedding theorems

Theorem [DS 2000]:

1. There is a truncation closed embedding $T : \mathcal{H}_{\mathcal{G},\exp} \longrightarrow \mathbb{T}$ that represents asymptotic expansion.
2. Every one-variable generalized power series with natural support contained in the image $\mathbb{T}_{\mathcal{G},\exp}$ is of the form $F\left(X^{1/d}\right)$, where F is multisummable in the positive real direction.

Theorem [Rolin, Servi, S 2024]:

1. There is a truncation closed embedding $T : \mathcal{H}_{\mathrm{an}^\omega,\exp} \longrightarrow \mathbb{T}$ that represents asymptotic expansion.
2. Every one-variable generalized power series with natural support contained in the image $\mathbb{T}_{\mathrm{an}^\omega,\exp}$ is convergent.

Corollary 1: $\Gamma\restriction_{(a,\infty)}$ is not definable in $\mathbb{R}_{\mathrm{an}^\omega,\exp}$, and $\zeta\restriction_{(a,\infty)}$ is not definable in $\mathbb{R}_{\mathcal{G},\exp}$, for any $a > 0$.

Lemma: The restriction of φ to $i(0,\infty)$ is not definable in $\mathbb{R}_{\mathcal{G},\exp}$.

Proof. If it is, then the restriction of $i\varphi(i/z)$ to the real line is definable as well. The latter has asymptotic series $i\Phi(iX)$, which is a real series. So by the embedding theorem, the latter series is multisummable in the positive real direction. Hence $\Phi(X)$ is multisummable in the direction i, a contradiction. □

Corollary 2: Let $F(X)$ be a multisummable series, let $\alpha \in [0, 2\pi)$ be a Stokes direction of F, and let f be a multisum of F. Then the restriction if f to $e^{i\alpha}(0,\infty)$ is not definable in $\mathbb{R}_{\mathcal{G},\exp}$.

Let $F(X)$ be a real 1-summable power series in the positive real direction, and let $f : S \longrightarrow \mathbb{C}$ be its multisum on the sector $S = \{z : |z| < R, |\arg z| < \alpha\}$, where $R > 0$ and $\alpha > \frac{\pi}{2}$.

Theorem [Padgett and S, preprint]:

1. For any $r < R$ and $\mu < \alpha - \frac{\pi}{2}$, the restriction of φ to the sector $S' := \{z : |z| < r, |\arg z| < \mu\}$ is definable in $\mathbb{R}_{\mathcal{G}}$.
2. Let $\gamma : (0, \infty) \longrightarrow \mathbb{C} \setminus (-\infty, 0)$ be a curve such that $\lim_{t\to\infty} \operatorname{re}\gamma(t) = -\infty$. Then $\varphi \circ \gamma$ is not definable in any o-minimal expansion of $\mathbb{R}_{\mathcal{G},\exp}$.

Corollary: The restriction of φ to any closed and bounded subsector of the right half-plane is definable in $\mathbb{R}_{\mathcal{G}}$, but the restriction to any left-unbounded set is not definable in any o-minimal expansion of $\mathbb{R}_{\mathcal{G},\exp}$.

Question: What about restrictions of φ to sectors $S = \left\{z : |z| < R, |\arg z| < \frac{\pi}{2}\right\}$?

The situation for convergent generalized power series is similar (but easier): Let $F(X)$ be a convergent generalized power series, and let f be its sum, defined on strip $S = \{x \in \mathbb{L} : |x| < R\}$, where $R > 0$ and $\mathbb{L}$ is the Riemann surface of *log*.

Theorem [PS, preprint]:

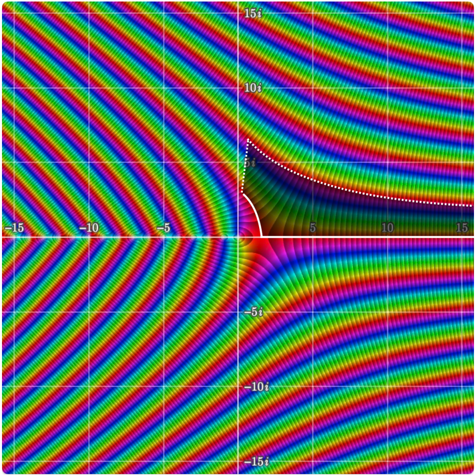
1. For any $\alpha > 0$, the restriction of f to $\{x \in S : |\arg x| < \alpha\}$ is definable in $\mathbb{R}_{\mathrm{an}^\omega}$
2. The restriction of f to any $\arg$-unbounded subset of S is not definable in any o-minimal expansion of $\mathbb{R}_{\mathrm{an}^\omega}$.

Corollary:

1. For any $t \in \mathbb{R}$ and $s > 0$, the restriction of ζ to the set $\{z \in \mathbb{C} : \operatorname{re} z > t, |\operatorname{im} z| < s\}$ is definable in $\mathbb{R}_{\mathrm{an}^\omega,\exp}$.
2. The restriction of ζ to any im-unbounded subset of $\{\operatorname{re} z > 2\}$ is not definable in any o-minimal expansion of $\mathbb{R}_{\mathrm{an}^\omega,\exp}$.

complex definability of Γ

Using Binet's second formula and our definability theorem for multisums, we get that the restriction of Γ to the following kind of domains is definable in $\mathbb{R}_{\mathcal{G},\exp}$:



more on Γ and ζ

In [RSS 2023], we define the notion of *multisummable generalized power series* (in the positive real direction).

$\mathcal{G}^* = (\mathcal{G}_n^*)_{n\in\mathbb{N}}$ class of all germs at 0^+ defined by generalized power series that are multisummable in the positive real direction. $\mathcal{G}_1^*$ contains both functions θ and φ defined earlier

$T : \mathcal{G}_n^* \longrightarrow \mathbb{R}[\![X_1, \dots, X_n]\!]$ the corresponding embedding (inverse of Borel-Laplace summation)

$\mathbb{R}_{\mathcal{G}^*}$ expansion of the real field by all restricted representatives of germs in $\mathcal{G}^*$. This expands both $\mathbb{R}_{\text{an}^\omega}$ and $\mathcal{R}_{\mathcal{G}}$

Theorem [RSS 2023]: The structure $(\mathbb{R}_{\mathcal{G}^*}, \exp)$ is o-minimal and defines both $\zeta\restriction_{(1,\infty)}$ and $\Gamma\restriction_{(0,\infty)}$. Moreover, a corresponding embedding into $\mathbb{T}$ also holds.

what's next?

Dulac's problem, which states that every real analytic vector field on the sphere has only finitely many limit cycles, was declared open again by Ilyashenko in 2022.

Fact: If there is an o-minimal structure in which all transition maps near elementary singularities of planar analytic vector fields are definable, then Dulac's problem follows.

Such a structure would have to define multisums as well as Ilyashenko's almost regular germs. One idea is to further extend the notion of multisummability by considering series of almost regular germs.

As a warmup, my student **Ilgwon Seo** is currently doing this for a subclass of Ilyashenko's almost regular germs.

Thank you!