On the canonical, fpqc and finite topologies: classical questions, new answers (and conversely)

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Up to a finite covering, a sequence of nested subvarieties of an affine algebraic variety just looks like a flag of vector spaces (Noether); understanding this "up to" is a primary motivation for a fine study of finite coverings.

The aim of this talk is to give a bird-eye view of some fundamendal questions about them, which took root in Algebraic Geometry (descent problems etc.), then motivated major trends in Commutative Algebra (F-singularities etc.), and recently found complete solutions using p-adic methods (perfectoids, prisms).

Rather than going into detail of the latter, the emphasis will be on synthesizing, from the geometric viewpoint, a rather scattered theme.

This talk is based on *joint work with Luisa Fiorot (Padova), to appear in Ann. Scuola Norm. Sup. Pisa*.

Base change and descent

$R \stackrel{\alpha}{\to} S$ homomorphism of commutative rings.

Base change functor: $\alpha^* = - \otimes_R S \;:\; Mod_R \to Mod_S$.

- α^* exact: α flat
- α^* faithful exact: α faithfully flat
- α^* faithful: α pure

(faithfully flat = flat + pure).

Fact: α pure $\Leftrightarrow$ α universally injective (i.e. remains injective after any base-change).

Base change and descent

Criteria:

• If α splits in Mod_R (i.e. R is a direct summand of the R-module S), then α is pure.

Proposition (Lazard, Fedder)

Converse is true if either
S/R is finitely presented,
or R is a complete Noetherian local ring.

Base change and descent

• If there exists an S-module M which is faithfully flat over R, then α is pure (Bourbaki).

Question (Ferrand)

Converse?

(solved in our paper...)

Base change and descent

“Reverse base-change": descent.

Descent data: S-module N + isomorphism

$$N \otimes_R S \xrightarrow{\phi} S \otimes_R N$$

satisfying the usual cocyle relations.

They form a category $DD(\alpha)$ (objects: (N, ϕ)), and α^* factors as:

$$Mod_R \xrightarrow{C_\alpha} DD(\alpha) \xrightarrow{\text{forget } \phi} Mod_S.$$

Grothendieck's descent:
α *faithfully flat* $\Rightarrow$ C_α *is an equivalence.*

Base change and descent

Grothendieck's descent:
α *faithfully flat* $\Rightarrow$ C_α *is an equivalence.*

Olivier's descent (70'): flatness is irrelevant!
α *pure* $\Leftrightarrow$ C_α *is an equivalence* $\Leftrightarrow$ C_α *is fully faithful.*

(This remarkable result was long overlooked, partly because it was published only as an announcement, in a CRAS note.
In our paper, we reconstruct the proof in detail, starting from a key lemma stated in that note as a hint.)

Topological interpretation of purity

No need to comment about the geometric meaning of flatness in Algebraic Geometry...

But what is the geometric meaning of purity?

This is best expressed in terms of Grothendieck topologies, as follows...

Topological interpretation of purity

$\mathcal{C}$: category with initial object and fibered products, e.g. Aff_k (affine schemes over a commutative ring k).

• Grothendieck pretopology: class of "covering families" $(Y_i \to X)$ in $\mathcal{C}$, containing all isomorphisms, stable by base change and by composition.
• Grothendieck topology: moreover, any family refined by a covering family is a covering family.

Ex: • fpqc pretopology on Aff_k: covering maps $Y \to X$ are faithfully flat maps.
• fpqc topology on Aff_k: $Y \to X$ is a covering map iff there exists $Z \to Y$ such that the composition $Z \to X$ is faithfully flat.
• effective descent topology on Aff_k: covering families: those for which the corresponding C_α's are equivalences.

Topological interpretation of purity

• Canonical topology on $\mathcal{C}$: finest Grothendieck topology s.t. all representable presheaves are sheaves
(for all covering families $(Y_i \to X)$,
$F(X) \to \Pi F(Y_i) \rightrightarrows \Pi F(Y_i \times_X Y_j)$ is exact).

Ex. • canonical topology on Aff_k:

Basic fact:

Spec $S \to$ Spec R is a canonical covering iff $R \to S$ is pure.

By Olivier's result ('70): on Aff_k, Canonical topology = Effective descent topology.

(On the other hand, the fpqc topology is strictly coarser than the canonical topology (Raynaud-Gruson '71)).

Topological interpretation of purity

Basic fact:

Spec $S \to$ Spec R is a canonical covering iff $R \to S$ is pure.

Issue: α pure $\Rightarrow \forall T, \quad T \stackrel{\alpha_T}{\to} S \otimes_R T \rightrightarrows S \otimes_R S \otimes_R T$ is exact.

Prove the analog in Mod_R: $M \stackrel{\alpha_M}{\to} S \otimes_R M \rightrightarrows S \otimes_R S \otimes_R M$, with double map $(\eta_1 = \alpha_{S \otimes_R M},\ \eta_2 = 1_S \otimes_R \alpha_M)$. Let $K \stackrel{\beta}{\to} S \otimes_R M$ be such that $\eta_1 \beta = \eta_2 \beta$ and look at

$$\begin{tikzcd}
 & K \arrow[dl, dotted] \arrow[dr, "\beta"] \arrow[ddl, "\beta"'] & \\
M \arrow[rr, "\alpha_M"] \arrow[d, "\alpha_M"'] & & S \otimes_R M \arrow[d, "1_S \otimes_R \alpha_M"] \\
S \otimes_R M \arrow[rr, "\alpha_{S \otimes_R M}"] \arrow[d, "\pi"'] & & S \otimes_R S \otimes_R M \arrow[d, "1_S \otimes \pi"] \\
\mathrm{Coker}\alpha_M \arrow[rr, hook, "\alpha_{\mathrm{Coker}\alpha_M}"] & & S \otimes_R \mathrm{Coker}\alpha_M .
\end{tikzcd}$$

Topological interpretation of purity

Remarks.

• this topological interpretation of purity allows to globalize to non-affine schemes. However, for a finite covering $Y \to X$ (i.e. a finite surjective map), $\mathcal{O}_X \to \mathcal{O}_Y$ need not split if X is not affine.

• For a long time, only subcanonical topologies were considered to be relevant to Algebraic/Arithmetic Geometry.
Voevodsky changed this viewpoint: some finer topologies are relevant because one can compute their points (local rings).
Another later motivation: (cohomological) descent properties.
- the v-topology ($Y \to X$ is a covering map if valuation rings of X lift to valuation rings of Y): v-descent plays a fundamental role in the geometric p-adic Langlands program (Fargues-Scholze).
- the arc-topology: defined similarly but for rank 1 valuations rings (Bhatt-Mathew): arc-descent plays a fundamental role in prismatic theory (Bhatt-Scholze).

Finite coverings

Back to: finite coverings = finite surjective morphisms.

Question: behaviour of finite coverings w.r.t. the canonical or fpqc topology?

• Examples of non-canonical finite coverings $Y \to X$ (with X affine Noetherian)?
- The normalization of any non-normal excellent scheme is an example (e.g. two crossing lines: $k[x, y]/(xy) \to k[x] \times k[y]$ is not pure, since its reduction mod $(x - y)$ sends $x \neq 0$ to 0).

- Ex. with X normal: $Y = \mathbb{A}^4_k \to X = \mathbb{A}^4_k/(\mathbb{Z}/4)$, where char $k = 2$ and $\mathbb{Z}/4$ acts cyclically on the variables.

Finite coverings

The point is that here X is not Cohen-Macaulay (M.J. Bertin '67), whereas a finite canonical covering Spec $S \to$ Spec R descends the Cohen-Macaulay property:
Indeed, in the following diagram, the vertical and the top horizontal arrows are injective since α is pure and S is Cohen-Macaulay

$$\begin{tikzcd}
\frac{S}{(x_1,\cdots,x_i)} \arrow[r, hook, "\cdot x_{i+1}"] & \frac{S}{(x_1,\cdots,x_i)} \\
\frac{R}{(x_1,\cdots,x_i)} \arrow[u, hook] \arrow[r, "\cdot x_{i+1}"] & \frac{R}{(x_1,\cdots,x_i)} \arrow[u, hook]
\end{tikzcd}.$$

finite coverings

Remark. No such example in char. 0. If char $k = 0$ and X is normal, any finite covering $Y \to X$ is canonical
(one may assume $R \subset S$ is an extension of integral domains, then $\mathrm{tr}_{S/R}/[S : R]$ gives a splitting).

Finite coverings

Going further, is there any example with X regular? No:

Theorem 1

Any finite covering of a regular scheme is canonical (hence an effective descent morphism).

Due to the above interpretation of purity, this amounts to a geometric translation of the direct summand conjecture (Hochster '69):

any finite extension S of a regular ring R splits in Mod_R,

proved by him in the presence of a base field, and by me in '16 in general, using perfectoid methods.

Finite coverings

• Examples of canonical finite coverings $Y \to X$ which are not coverings for the fpqc topology?

- Frobenius-type examples: M finite R-module, $b : Sym^2 M \to R$, associative: $\ell b(m, n) = b(\ell, m)n$. Then $Ann_R M$ kills $b(m, m)$, hence $b(m, m) \in J := Ann_R(Ann_R M)$.
$S := R \oplus M,\ (r, m)(r', m') = (rr' + b(m, m'), rm' + r'm)$.

Proposition

If $b(m, m) \notin J^2$, then the finite canonical covering $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is not a covering for the fpqc topology.

e.g. $R = \mathbb{Z}/4,\ M = \mathbb{Z}/2,\ J = (2),\ b(1, 1) = 2.$

Finite coverings

- Can one find a normal example? Yes: the quadric cone.
k: field of char. $\neq 2$, $S = k[x, y]$, $\mathbb{Z}/2$-action $(x, y) \mapsto (-x, -y)$,
$S := k[x, y]^+ \oplus k[x, y]^-$.
Thus $R := k[x, y]^+ = k[x^2, xy, y^2] \to S$ is pure.

Proposition

Spec $S \to$ Spec R is not a covering for the fpqc topology.
Moreover, no S-module is faithfully flat over R.

(Negative answer to Ferrand's question, using recent work of Bhatt-Iyengar-Ma).

Finite coverings

Going further, is there any example with X regular? No:

Theorem 2

Any finite covering of a regular scheme is a covering for the fpqc topology.

Remarks. 1) $Y \to X$ is a covering for the fpqc *pre*topology iff Y is Cohen-Macaulay.
2) Thm 2 is much stronger than Thm 1: e.g. unlike Thm 1, it is quite non-trivial in char. 0 (reduction to char. p and ultraproducts).
3) Some finite coverings of $\mathbb{A}^3_k$ are not coverings for the fppf topology (in general, Z is not expected to be of finite type, nor even Noetherian).

Finite coverings

Thm. 2 is a geometric translation of the existence of "big" Cohen-Macaulay algebras, proved by Hochster-Huneke in the presence of a base field, by me in general ('16).
Reduce to S complete local, finite extension of regular R (Cohen): a (not necessarily Noetherian) S-algebra T is (big) Cohen-Macaulay if every secant sequence in S becomes regular in T. This is equivalent to the faithfull flatness of T over R.

Finite coverings

• My construction ('16). For simplicity, take $R = \mathbb{Z}_p[[x_2, \cdots, x_n]]$, and S a reduced, finite extension of R.
Extract p-roots of parameters:
$R_\infty = \mathbb{Z}_p[p^{1/p^\infty}][[x_2^{1/p^\infty}, \cdots, x_n^{1/p^\infty}]]\hat{.}$
Let $g \in R$ be such that $S[1/pg]$ is etale over $R[1/pg]$. Extract p-roots and take integral closure in the resulting algebra with p inverted:
$R_{\infty\infty} = i.c.(R_\infty[g^{1/p^\infty}], R_\infty[g^{1/p^\infty}, 1/p])\hat{.}$
These “big" algebras have the perfectoid property:
$A/p^{1/p} \overset{x \mapsto x^p}{\to} A/p$ is an isomorphism.

Finite coverings

$S_{\infty\infty} = i.c.(S \otimes_R R_{\infty\infty}, S \otimes_R R_{\infty\infty}[1/pg])$.

A "perfectoid Abhyankar lemma" then implies that $S_{\infty\infty}$ is a $(pg)^{1/p^\infty}$-“almost" Cohen-Macaulay S-algebra.

To get rid of “almost", an earlier trick due to Hochster, or a later trick due to Gabber, transforms $S_{\infty\infty}$ into a genuine (big) Cohen-Macaulay S-algebra T, i.e. a T which is faithfully flat over R: e.g.

$$T = \Sigma^{-1}(S_{\infty\infty}^{\mathbb{N}}/S_{\infty\infty}^{(\mathbb{N})})$$

(Σ = multiplicative system of $((pg)^{\varepsilon_n})$, $\varepsilon_n \to 0$ in $\mathbb{N}[1/p]$).
Moreover, T is perfectoid.

Finite coverings

• Bhatt's construction ('21): (takes into account all finite domain extensions S at the same time)

R: excellent regular domain with $p \in rad\, R$,
R^+: integral closure of R in an algebraic closure of Frac R.
Then $\widehat{R^+}$ is faithfully flat over R.

(in char. p: due to Hochster-Huneke).

This uses a new p-adic Riemann-Hilbert functor (Bhatt-Lurie) to get the "almost" result, then prismatic techniques (where Frobenii are at disposal) to get rid of "almost".

Motto: *"in some situations, local or coherent cohomology classes can be killed by passing to finite coverings"*.

Frobenius

• In char. p, the Frobenius map $F_X : X \to X$ is faithfully flat iff X is regular (Kunz).
Enhancement (from the fpqc pretopology to the fpqc topology):

Proposition

F_X is a covering for the fpqc topology iff X is regular.

Frobenius

Remarks. 1) F_X is often a finite covering (e.g. for schemes of finite type over a perfect field).

2) Whether F_X is a canonical covering (*F-purity*: Frobenius splitting) was an influential question in the study of singularities in char. p in the 80's, 90's ("F-singularity theory").

Related notion: *strongly F-regular* (Noetherian domain) S:
$\forall\, s \in S, \exists\, e, \exists\, S^{1/p^e} \to S$, s.t. $s^{1/p^e} \mapsto 1$.
F-pure and strongly F-regular singularities are char. p analogs of log-canonical and log-terminal singularities in the MMP.

Frobenius

These have now analogs in mixed char. (Ma, Schwede), based on Thm. 2.

Ma and Schwede remark that for a resolution of singularities $g : Z \to Y = \mathrm{Spec}\, S$ in char. 0, Grauert-Riemenschneider + local duality implies $\mathbb{H}^i_y(Rg_*\mathcal{O}_Z) = 0$ for $i < \dim Y$, hence $Rg_*\mathcal{O}_Z$ looks like a Cohen-Macaulay S-algebra... except that it is an algebra object in the derived category.

This supports their strategy to replace $Rg_*\mathcal{O}_Z$ by a big Cohen-Macaulay S-algebra T to study singularities in mixed char.

Splinters

• Back to: finite coverings vs. canonical topology.

A Noetherian affine scheme $X = \operatorname{Spec} R$ is a splinter is any finite covering of X is canonical

- equivalently: if any finite extension $R \to S$ splits in Mod_R (whence the name).

- Any splinter is normal.
- Thm. 1 says that any regular X is a splinter.

Splinters

Are there any other? Yes.

- In char. 0, any normal X is a splinter (by the tr/deg trick)

- In char p, any strongly regular X is a splinter (converse true in the Gorenstein case, and conjecturally in general)

e.g. the quadric cone Spec $k[x^2, xy, y^2]$ in char. $\neq 2$ is a splinter.

- Splinters in positive or mixed char. are Cohen-Macaulay

(a straightforward consequence of the Hochster-Huneke-Bhatt theorem about $\widehat{R^+}$).

- X is a splinter if there exists a canonical covering Y which is regular (this is a main supply of splinters; the proof uses a “weakly functorial" version of Thm. 2)

Splinters

• What if we replace the canonical topology by the fpqc topology ("fpqc analogs of splinters")?

Conjecture: only regular X are fpqc analogs of splinters.

Evidence: - true in char. p in the F-finite case (a straightforward consequence of our enhancement of Kunz' theorem),

- regularity descends along maps which are coverings for the fpqc topology, hence the "main supply of splinters" cannot provide any non-regular fpqc analog of splinters.

Splinters

p.s. in EGA, we find many properties of schemes and morphisms which descend along faithfully flat morphisms (i.e. w.r.t to the fpqc *pre*topology).

This suggests to revisit which properties descend w.r.t. the fpqc topology (not only for the fpqc pretopology), ex: regularity.
While this issue could have been addressed in the 60's, it brings us back, after a long detour, to fundamental problems about base-change and descent we started with.