Cut-off Phenomena and Cyclic Group

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Cut-off Phenomenon

Example 1 (Alous, Diaconis 1986)

Take a deck of n cards. At each step, remove the top card and insert it randomly in the deck. When the card initially on the bottom reaches the top, it is randomly inserted into the deck and hence all $n!$ arrangements are equally likely.

Cut-off Phenomenon

Example 1 *continued*

The bottom card stays on the bottom of the deck until a card is inserted below it, which initially has probability $\frac{1}{n}$ and is expected to take n steps. When there are k cards below the original bottom card, there is probability $\frac{k}{n}$ of inserting a card below the original and thus this is expected to take $\frac{n}{k}$ steps. Hence it follows inductively that the number of steps for shuffling is

$$n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n} \approx n \log n$$

Formalizing "Distance to Random"

Total–Variation Distance

The total-variance distance between two probability distributions μ and ν on finite state space Ω is defined in the following way:

$$\|\mu - \nu\|_{TV} := \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| \tag{1}$$

Distance to stationary distribution

For a Markov chain with transition kernel P and stationary law π, we define its distance to the stationary distribution π at time t to be

$$d(t) := \left\| P^t(x) \; - \; \pi(x) \right\|_{\mathrm{TV}}$$

One can verify that $d_x(t) \in [0, 1]$.

The Definition of Cut-off

Cut-off Time and Window

A family of chains $\{P_n\}$ on state-spaces Ω_n exhibits *cut-off* if there exist

$$t_n \quad \text{(cut-off time)} \quad \text{and} \quad w_n = o(t_n) \quad \text{(window)}$$

such that:

$$\lim_{n\to\infty} d_n\big(t_n - c\,w_n\big) = 1, \quad \lim_{n\to\infty} d_n\big(t_n + c\,w_n\big) = 0, \quad \forall\, c > 0.$$

- *Before* t_n: chain is far from stationarity (ordered and structured).
- *After* t_n: chain is essentially mixed (random).
- Transition happens over negligible window w_n.

Regular graph and cutoff

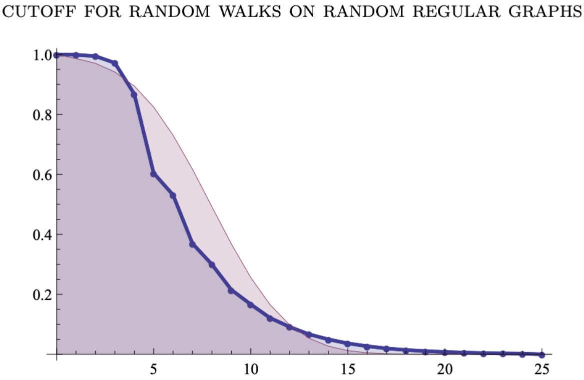

Figure 1: Distance from stationary distribution along time for the simple random walk on a random 6-regular graph on $n = 5000$ vertices

Cay(Z_n, S)

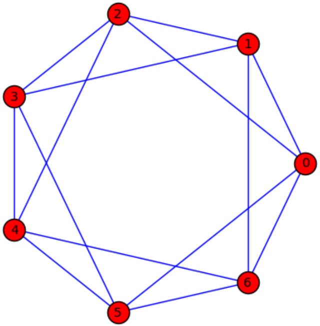

Figure 2: Cay(Z_7, $\{1, 2\}$)

Theorem (Diaconis, Saloff-Coste 1994)

As $d \to \infty$, simple random walk (**SRW**) displays cutoff.

Finite generating sets

Theorem (Lubetzky and Sly 2008)

Let $G \sim \mathcal{G}(n, d)$ be a random regular graph with $d \geq 3$ fixed. Then, ***whp****, the simple random walk on G exhibits cutoff behavior.*

Idea

What if we choose the generating set S finite, e.g.

$$S = [k] = \{1, 2, \ldots, k\}?$$

Testing Cutoff

Definition

Write the eigenvalues of the transition matrix as $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\lambda_{gap} \;:=\; 1 - \lambda_1.$$

The mixing time $t_{mix}(\epsilon)$ is minimum t such that

$$d(t) \;\leq\; \epsilon.$$

Theorem (Product Condition, Peres 2004)

If

$$\lim_{n\to\infty} \lambda_{gap} t_{mix}(\epsilon) < \infty$$

then the system cannot display cutoff behavior.

The eigenvalue gap I

Theorem (Babai 1979)

Suppose $\chi_i(g)$ is a character of the group G. The spectrum of a Cayley color graph is given by the following formula,

$$\lambda_j = \sum_{g \in G} \alpha(g)\chi_j(g) \tag{2}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of the adjacency matrix of the color graph.

Let G be an unordered Cayley graph associated to group Z_n and a generating set $[k]$, i.e. $G = \text{Cay}(Z_n, [k])$. Then, the adjacency matrix A_G has the spectrum

$$\lambda_j = \frac{\csc(\theta_j/2)\sin\left((k+\frac{1}{2})\theta_j\right) - 1)}{2k} \tag{3}$$

The eigenvalue gap II

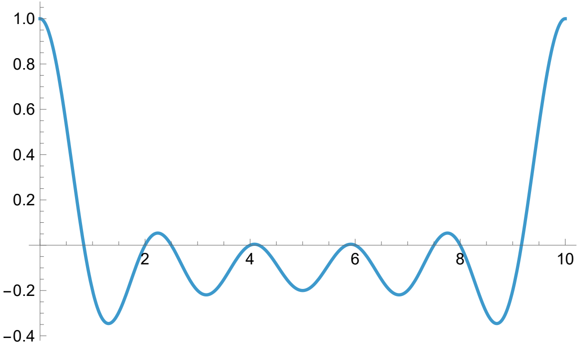

Figure 3: Eigenvalues for $n = 10, S = [5]$

The eigenvalue gap III

Corollary

Suppose T is the transition matrix of the random walk over $(Z_n, [k])$ for fixed k. The largest nontrivial eigenvalue λ_1 approaches 1 as n grows to ∞.

Conclusion I

Theorem

The family of graphs $(Z_n, [k])$ *for fixed* k *and increasing* n *does not satisfy the product condition, i.e. the following limit holds.*

$$\lambda_{gap} t_{mix}(\epsilon) = \log\left(\frac{2}{\epsilon\pi}\right) \tag{4}$$

Therefore, a random walk over $(Z_n, [k])$ *does not display cutoff behavior.*

Proof Sketch.

For large n, the eigenvalue gap is inverse quadratic w.r.t. n, and the mixing time is quadratic w.r.t. n. □

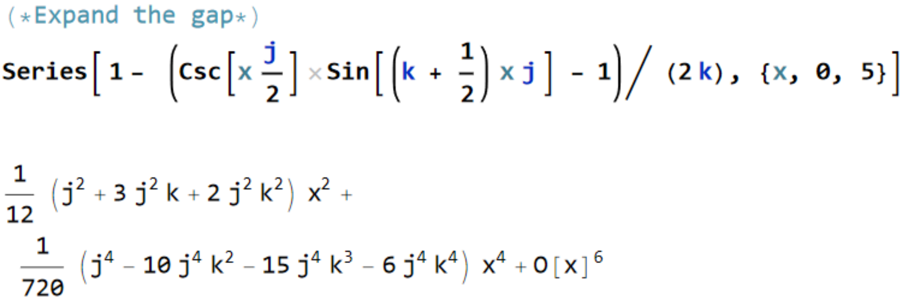

Figure 4: Taylor series expansion of $\lambda_{gap} = 1 - \lambda_2$, $x = \frac{2\pi}{n}$

Conclusion III

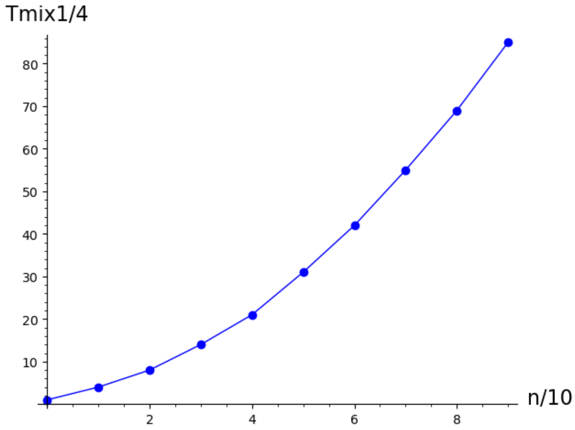

Figure 5: Computational result for mixing time given $S = [5]$

Future Work

Here are some future directions that we are interested in exploring:

- Cayley graphs on different generating set S
- Non-backtracking random walk

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Thank you!