Loop Dynamics and a Geometric Solution of Planar QCD

Lecture II: Momentum Loop Space and the Failure of Taylor–Magnus

Alexander Migdal

Institute for Advanced Study

February 21, 2026

Discussion map: choose a fork

We can go in (at least) three directions today. Please interrupt and pick:

Fork A: Momentum Space

- The physical quark loop measure
- Fourier transform of $W[C]$
- Eliminating $\delta^{(4)}(x-y)$

Fork B: Algebraic MLE

- Magnus forms and Shuffle Ideal
- Algebraic recurrence relations
- Exact solutions up to $\mathcal{O}(P'^7)$

Fork C: The Catastrophe

- The breakdown at $\mathcal{W}^{(8)}$
- Vector vs. Scalar mismatch
- Why naive string theory fails

(Let's walk through the finite algebraic structure, and you'll see why it forces us into a non-perturbative regime.)

Regularization, Renormalizability, and Lattice QCD

- **The Standard Approach:** Lattice QCD provides a rigorous microscopic definition.
- **The Cost:** It explicitly breaks the continuous symmetries of 4D Euclidean space ($O(4)$ rotations) and subtle topological structures like **Hodge duality**.
- **Exact geometry requires the continuum:** Exact instanton solutions or Hodge-dual minimal surfaces simply do not exist on a discrete grid; they only emerge in the continuum limit.
- **Our Strategy:** Keep the theory regularized at the level of the Wilson loop $W[C]$ by a finite fermion mass m (which acts as a geometric UV cutoff).

Shifting the Paradigm: From Coordinates to Momentum

- **Is $W[C]$ physical?** The coordinate-space Wilson loop $W[C]$ is an intermediate, off-shell quantity.
- **Physical observables** involve path integrals over quark trajectories governed by a Brownian measure (an infinite density of cusps!).
- Attempting to renormalize $W[C]$ contour-by-contour for a smooth curve is physically artificial.
- **The Solution:** Transition to **Momentum Loop Space**.

$$W[P] = \int \mathcal{D}C \, W[C] \exp\left(\mathrm{i} \oint dx \, P_\mu(x) \dot{C}_\mu(x)\right); \tag{1}$$

$$\mathcal{D}C = \delta^4\left(\int_0^{2\pi} \dot{C} d\theta\right) \prod_{\theta=0}^{2\pi} d^4 \dot{C}(\theta) \tag{2}$$

- By integrating over coordinate loops, we integrate out the geometric singularities (cusps, contact terms) natively!

Quark Loop Amplitudes in Phase Space

- In momentum loop space, physical observables are clear and finite.
- The amplitude for the propagation of a free Dirac particle around the phase space loop:

$$\mathcal{K}\left[P\right] = \operatorname{tr} \hat{P} \exp\left(-\int_0^\infty dt \left(\mathrm{i}\gamma_\mu P_\mu(t) + m_q\right)\right) \tag{3}$$

- The full scattering amplitude with injected external momenta q_k:

$$A[q_1, \dots, q_n] \propto \int \prod dt_k \int \mathcal{D}P\, W[P + Q]\, \mathcal{K}\left[P\right] \tag{4}$$

where $Q(t) = \sum_k q_k \Theta(t - t_k)$ and $\sum_k q_k = 0$

Feynman Wheel

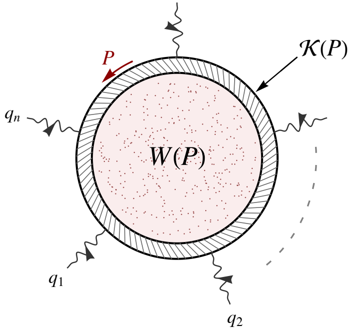

Figure: The momentum loop amplitude, $A(q_1, \ldots q_n)$ with the inside part of the wheel corresponding to Momentum loop $W[P]$, and the outer rim corresponding to Dirac path amplitude $\mathcal{K}[P]$. The paths $P(t)$ are random, interacting with inner geometry of the string surface represented by amplitude $W[P]$.

Factorization of the Measure in Momentum Space

- How does the $\delta^{(4)}(x-y)$ contact term from the MM equation disappear?
- Consider the linear functional measure in loop space. For a self-intersecting loop $C(\theta_1) = C(\theta_2)$, the measure natively factorizes!

$$\delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \mathcal{D}C = \left(\delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \prod_{\theta'=\theta_1}^{\theta_2} d^4\dot{C}(\theta')\right)\left(\delta^4\left(\int_{\theta_2}^{\theta_1+2\pi} \dot{C} d\theta\right) \prod_{\theta'=\theta_2}^{\theta_1+2\pi} d^4\dot{C}(\theta')\right)$$

$$\implies \delta^4\left(\int_{\theta_1}^{\theta_2} \dot{C} d\theta\right) \mathcal{D}C = \mathcal{D}C_{12}\mathcal{D}C_{21} \tag{5}$$

- The singular contact constraint $\delta^{(4)}$ is strictly absorbed by the measure of the sub-loops.

The Momentum Loop Equation (MLE)

- Under this Fourier transform, the coordinate Makeenko-Migdal equation:

$$\mathcal{L}_\nu(W[C]) = \lambda \int_0^{2\pi} d\theta\, C'_\nu(\theta)\delta^4(C(0) - C(\theta))W[C_{0,\theta}]W[C_{\theta,2\pi}] \tag{6}$$

becomes the **Momentum Loop Equation**:

$$\hat{L}_\nu[P]W[P] = \int_0^{2\pi} d\theta\, \frac{\delta}{\delta P_\nu(\theta)} \left(W[P_{0,\theta}]W[P_{\theta,2\pi}]\right) \tag{7}$$

- **Left Side (Kinematics):** Algebraic multiplication by a kinematic tensor $\hat{L}_\nu[P]$.
- **Right Side (Dynamics):** Functional derivative acting on the convolution of two sub-loops.
- **Crucial feature:** *No delta functions. No cusp logarithmic divergences.* The equation is purely algebraic-differential and completely finite.

The Kinematic Tensor and the Shuffle Ideal

- The loop diffusion operator $\mathcal{L}_\nu = \partial_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(0)}$ corresponds to a specific trilinear product:

$$\hat{L}_\nu[P] = T^{\alpha\beta\gamma}{}_\nu \Omega^{(3)}_{\alpha\beta\gamma}[P], \tag{8}$$

$$\Omega^{(3)}_{\alpha\beta\gamma}[P] = \int_0^{2\pi} d\tau_1 P'_\alpha(\tau_1) \int_0^{\tau_1} d\tau_2 P'_\beta(\tau_2) \int_0^{\tau_2} d\tau_3 P'_\gamma(\tau_3) \tag{9}$$

  Here $\Omega^{(3)}$ is the third-order Magnus form (ordered triple integral of P').

- The tensor $T^{\alpha\beta\gamma}{}_\nu$ implements the exact Dynkin projector onto the nested double commutator $T \cdot (D \otimes D \otimes D) \sim [\hat{D}, [\hat{D}, \hat{D}]]$:

$$T^{\alpha\beta\gamma}{}_\nu = \delta_{\alpha\beta}\delta_{\gamma\nu} + \delta_{\gamma\beta}\delta_{\alpha\nu} - 2\delta_{\alpha\gamma}\delta_{\beta\nu} \tag{10}$$

- By Ree's Theorem for Free Lie Algebras, this tensor identically annihilates the **symmetric shuffle ideal** (symmetrized path integral) associated with the path-ordered integral, leaving just $\Omega^{(3)}$.

Algebraic Recurrence for the MLE

- We construct the functional Taylor series by expanding $W[P]$ in parametric-invariant **Magnus forms** $\Omega^{(n)}$:

$$W[P] = \operatorname{tr}\left\{\hat{T}\exp\left(\left(\int d\theta \hat{X}_\mu P'_\mu(\theta)\right)\right)\right\} = \sum_n \mathcal{W}^{(n)}_{\alpha_1\ldots\alpha_n}\Omega^{(n)}_{\alpha_1\ldots\alpha_n}[P] \tag{11}$$

- The constant operator $\hat{X}_\mu$ has the meaning of the position of quark. The MLE provides recurrent equations between multiple traces $\operatorname{tr}\hat{X}^{\otimes n}$.
- When the functional derivative acts, it brings down an exact commutator insertion:

$$\frac{\delta}{\delta P_\mu(a)}\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\} = P'_\nu(a)\operatorname{tr}\left\{[\hat{X}_\mu,\hat{X}_\nu]\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\}; \tag{12}$$

$$\frac{\delta}{\delta P_\mu(b)}\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)\right\} = -P'_\nu(b)\operatorname{tr}\left\{\hat{T}\exp\left(\left(\int_a^b \hat{X}_\alpha P'_\alpha\right)\right)[\hat{X}_\mu,\hat{X}_\nu]\right\} \tag{13}$$

- We dynamically equate LHS kinematic shuffle products against RHS discrete commutators. The MLE collapses into a finite, recursive combinatorial algebra!

The Magnus forms MLE

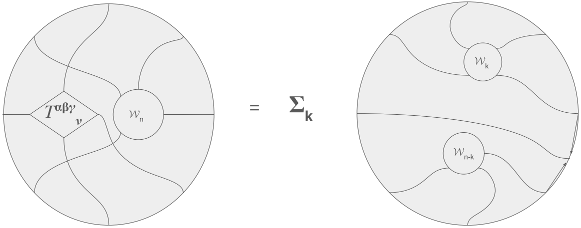

Figure: The recurrent equation for the coefficient tensor parameters $\mathcal{W}^n$ in the Magnus expansion. The external circle represents the unit circle where the angular parameter belong. The inner round blobs with wavy legs landing on a unit circle represent the $\mathcal{W}^n$ tensors, the rhombus with four wavy legs represents the 4-tensor T . Two arrows on the right side correspond to functional derivatives $\frac{\delta}{\delta P_\nu}$ bringing commutators by momentum area derivatives.

Topological Closure and Exact Solutions

- Because the momentum loop is physically closed, $\Omega^{(1)} = \oint P' d\theta = 0$.
- The functional derivative on the RHS splits the loop into two *open* subloops. We analytically enforce their closure by introducing a boundary gap:

$$P'_{closed}(\theta') = P'(\theta') + \Delta P \delta(\theta' - \theta_0) \tag{14}$$

- This subtracts the product $\Omega^{(1)}\Omega^{(n-1)}$ (interleaved operators) from the n-th order form.
- **Order $\mathcal{O}(P'^3) \implies \mathcal{W}^{(4)}$:** Solved exactly.
- **Order $\mathcal{O}(P'^5) \implies \mathcal{W}^{(6)}$:** Solved exactly. The geometric stress cascading from the lower-order loop derivatives is exactly absorbed by the independent Kronecker pairings.

Exact Algebraic Solutions up to $\mathcal{O}(P'^5)$

- We solve the algebraic MLE by expanding $W[P]$ in the full $(n-1)!!$ general non-cyclic Wick-contraction basis.
- To display the tensors compactly, we use the shorthand $\delta_{ij} \equiv \delta_{\mu_i \mu_j}$.

Order $\mathcal{W}^{(4)}$ (from $\mathcal{O}(P'^3)$):

$$\mathcal{W}^{(4)}_{1234} = \frac{1}{6}\delta_{13}\delta_{24} + c_{4,1}\left(\delta_{12}\delta_{34} + \delta_{13}\delta_{24} + \delta_{14}\delta_{23}\right) \tag{15}$$

Order $\mathcal{W}^{(6)}$ (from $\mathcal{O}(P'^5)$):

$$\begin{aligned}
\mathcal{W}^{(6)}_{123456} = {} & c_{6,1}\left(\delta_{12}\delta_{34}\delta_{56} + \delta_{16}\delta_{23}\delta_{45}\right) \\
& + c_{6,2}\big(\delta_{13}\delta_{24}\delta_{56} + \delta_{12}\delta_{35}\delta_{46} + \delta_{15}\delta_{23}\delta_{46} \\
& \qquad + \delta_{13}\delta_{26}\delta_{45} + \delta_{16}\delta_{24}\delta_{35} + \delta_{15}\delta_{26}\delta_{34}\big) \\
& + c_{6,3}\left(\delta_{14}\delta_{23}\delta_{56} + \delta_{12}\delta_{36}\delta_{45} + \delta_{16}\delta_{25}\delta_{34}\right) \\
& + \left(2c_{6,2} - \frac{1}{2}c_{6,1} - \frac{1}{2}c_{6,3}\right)\left(\delta_{13}\delta_{25}\delta_{46} + \delta_{14}\delta_{26}\delta_{35} + \delta_{15}\delta_{24}\delta_{36}\right)
\end{aligned}$$

The Breakdown: Mathematical Contradiction at $\mathcal{W}^{(8)}$

- The geometric stress rigidly locks the coefficients, perfectly absorbing the constraints.
- **At $\mathcal{W}^{(8)}$, this linear absorption violently fails!**
- **The Source Term:** The RHS generates a massive, inhomogeneous geometric stress sourced by the non-linear cross-terms of the massive 4-point functions: $\mathcal{W}^{(4)} \times \mathcal{W}^{(4)}$.
- **The Trap:** The loop derivative acts effectively as a commutator: $\mathcal{D}(\mathcal{W}^{(8)}) \approx \mathcal{W}^{(8)}_{\nu\alpha_1\dots} - \mathcal{W}^{(8)}_{\alpha_1\nu\dots}$. This strictly annihilates any symmetric index structures in the Ansatz.
- **The Contradiction:** Even using the most general $(n-1)!! = 105$ non-cyclic Kronecker pairings, the surviving matrix rank of the variables is crushed. It is *algebraically impossible* to absorb the rigid, symmetric cross-term stress generated by $\mathcal{W}^{(4)} \times \mathcal{W}^{(4)}$.

The Root Cause: Vector vs. Scalar Mismatch

Why does the general expansion fail?

The fundamental root of this non-analyticity is that the loop equation is inherently a **vector** equation:

$$\hat{L}_\nu W = \int \frac{\delta}{\delta P_\nu}(W \times W) \tag{17}$$

But it governs a **scalar** functional $W[P]$.

- The number of independent tensor structures available for a scalar functional grows strictly slower ($\sim 4\times$ slower in $d = 4$) than those required for a free-index vector equation.
- At some critical order ($\mathcal{W}^{(8)}$), there will inevitably be more vector equations than available scalar parameters. The system becomes fatally overdetermined.

The Consequence for Naive String Theories

- The Loop Equation descends directly from the vector Yang–Mills equations of motion, dictating that a vector equation must be satisfied at every fixed point on the contour.
- Resolving this overdetermined system demands a deep mathematical "conspiracy" among the internal degrees of freedom within $W[P]$.
- **A Profound Corollary:** This exact vector-versus-scalar mismatch is fatal to all naive bosonic string theories of QCD!
- The Nambu-Goto string Laplace operator is a *scalar*. A scalar area-derivative constraint fundamentally cannot capture the full vector loop equation without generating an anomaly.

Physical Meaning: The Non-Analytic Vacuum

- The geometric contradiction at the 8th order rigorously proves that the exact solution $W[P]$ is **not an analytic functional** of the bounding loop $P(\theta)$.
- It cannot be indefinitely expanded in a continuous Taylor–Magnus series.
- **The Elfin Solution:** Our Elfin theory natively provides the required conspiracy! By placing Majorana fermions on a rigid Hodge-dual minimal surface, the internal degrees of freedom identically satisfy the vector equations across all components.
- **Ultra-local Integrals:** The integration over the momentum loops is not dominated by smooth, small P variations. It factorizes point-by-point. The true continuum solution is inherently non-perturbative.

Summary of Lecture II

1. **Momentum Space Cures the UV:** Transitioning to $W[P]$ integrates out cusp and contact singularities, yielding a finite algebraic MLE.
2. **Magnus Expansion is Finite but Limited:** The system is analytically solvable up to 6th order, proving the equations are singularity-free in the UV.
3. **The Vector/Scalar Mismatch:** The scalar nature of $W[P]$ cannot satisfy the vector nature of the Yang–Mills loop equations perturbatively at the 8th order.
4. **Naive Strings Fail:** Bosonic string Laplacians are scalars; they lack the geometric degrees of freedom to satisfy the vector MLE.

Next Lecture:

How the Elfin/Twistor theory provides the missing "conspiracy" via Hodge Duality and Majorana fermions on a rigid minimal surface.

Addendum: Questions & Answers

From the extended discussion during Lecture II

Q&A 1: Does the Trace Ansatz Restrict Generality?

Question raised during the seminar:

Is the ansatz $W[P] = \operatorname{tr} \hat{T} \exp(() \, \mathrm{i} \int \hat{X}_\mu dP_\mu)$ the most general one? Doesn't a trace of finite matrices impose polynomial trace identities (like Cayley-Hamilton) that artificially restrict the Magnus expansion tensors $\mathcal{W}^{(n)}$?

- **Answer:** If $\hat{X}_\mu$ were finite $N \times N$ matrices, yes, the finite trace identities would fatally restrict the expansion.
- However, in the $N_c \to \infty$ limit, the Master Field operates in an **infinite-dimensional** Hilbert space.
- As shown rigorously in *(Migdal, 1994, "Second Quantization of the Wilson Loop")*, the true non-perturbative Fock space of the string endpoint is the space of **"words"** generated by $d = 4$ Cuntz algebra operators: $a_\mu a_\nu^\dagger = \delta_{\mu\nu}$.
- In the infinite-dimensional free Cuntz algebra, **there are zero trace identities.**

Q&A 2: The Single Universal Operator

- **Follow-up:** *But to match independent $\mathcal{W}^{(n)}$ tensors at every order n, wouldn't you need an n-dependent operator or n-dependent Hilbert spaces?*
- **Answer:** No. A single, universal operator $\hat{X}_\mu$ is sufficient.
- The exact construction *(Migdal, 1994)* represents the position operator as an infinite cascading sum of Cuntz creation operators:

$$\hat{X}_\mu = a_\mu + \sum_{k=1}^{\infty} Q_{\mu,\mu_1\dots\mu_k} a^\dagger_{\mu_1} \dots a^\dagger_{\mu_k} \tag{18}$$

- The single operator $\hat{X}_\mu$ contains an infinite tower of independent parameters (the planar connected moments Q).
- Therefore, the trace of this path-ordered exponential is **completely unrestricted** and exhaustively spans the functional space of any closed loop.

Q&A 3: Can the MLE be Absorbed into a Closed Algebra?

Question:

The Right-Hand Side splits the trace. This is equivalent to a non-linear operator variation $\delta \hat{X}_\mu = \epsilon_\nu([\hat{X}_\mu, \hat{X}_\nu]|0\rangle\langle 0| + |0\rangle\langle 0|[\hat{X}_\mu, \hat{X}_\nu])$*. Could the entire MLE just be the invariance of the Cuntz algebra under some non-linear symmetry generated by* $[\hat{X}_\mu, \hat{\Gamma}_\lambda]$*?*

- **Answer:** This beautiful idea works for the **scalar** (projected) loop equations *(Migdal, 1996)*. The RHS loop-splitting is indeed a perfect non-linear automorphism of the 1D Cuntz algebra.
- **But it fails for the exact Vector MLE.** The exact Left-Hand Side is a macroscopic geometric stress driven by a 3rd-order Magnus form (a shuffle product):

$$\hat{L}_\nu W[P] = T^{\alpha\beta\gamma}{}_\nu \Omega^{(3)}_{\alpha\beta\gamma}[P] \times W[P] \tag{19}$$

- A purely internal operator variation acts as a degree-1 derivation. It fundamentally **cannot** reproduce a degree-3 geometric shuffle product for an arbitrary path $P(\theta)$.

Q&A 4: The Necessity of 4D Geometry

- **The Physical Meaning:** Planar QCD **cannot** be reduced to a purely 1-Dimensional Topological Field Theory of words and letters.
- While the Master Field theorem guarantees $W[P]$ can be formally represented by $\operatorname{tr}\hat{T}\exp\left(()\, i\int \hat{X}dP\right)$, the 1D internal space of the Cuntz operators fundamentally lacks the geometric degrees of freedom to represent the 2D spatial stress of the true Yang-Mills vector equation.
- The Vector MLE is strictly incompatible with a closed 1D algebra for $\hat{X}$. This is the exact, rigorous reason why the Taylor-Magnus expansion violently fails at $\mathcal{W}^{(8)}$.
- **The Way Out:** There must be specific, 4D continuous geometry involved to absorb this stress. As we will see in Lecture III, this geometry is the **Hodge-dual minimal surface**.