Loop Dynamics and a Geometric Solution of Planar QCD

Lecture I: Loop Calculus and the Coordinate Space Dilemma

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Series Overview:

1. **Loop Calculus & The MM Equation:** The generation of planar graphs and the singularity catastrophe.
2. **Momentum Loop Space:** Finite algebraic structure and off-shell recursion.
3. **The Hodge-Dual Minimal Surface:** Zero modes, additivity, and the area law.
4. **The Fermi String:** The “Elfin” determinant, phase space path integral, and Analytic Twistor String.

The 50-Year Challenge

- **The Premise:** In the limit $N_c \to \infty$, QCD should correspond to a string theory (planar diagrams dominate).
- **The Object:** The Wilson Loop functional:

$$W[C] = \frac{1}{N_c} \left\langle \text{tr}\ \hat{P} \exp\left(\oint_C dx_\mu A_\mu(x) \right) \right\rangle \tag{1}$$

- **The Goal:** Find a string effective action $S_{eff}[C]$ such that:

$$W[C] = \int \mathcal{D}X \exp\left(-S_{eff}[X]\right) \tag{2}$$

- **The Obstacle:** For decades, the string dual remained elusive due to the **ultraviolet singularities** of $W[C]$ in coordinate space:
  - Cusp singularities.
  - Perimeter divergences.
  - Ill-defined loop equation at self-intersection points.

Wilson loop and physical amplitudes

- **Path integral for $\bar{q}q$ currents**.
  The observable current-current correlator is related to Wilson loop as path integral over quark velocity $v = C'$

$$\langle \bar{q}q\bar{q}q \rangle \propto \int d^4k \int ds_1 ds_2 \int DvW[C] \exp\left(\mathrm{i}q \int_{s_1}^{s_2} dtv + \oint dt(\mathrm{i}kv - v^2/2) \right) \quad (3)$$

- **Perturbation theory:**
  With

$$W = 1 - \lambda \int \frac{d^4p}{p^2} \int ds_1 v(s_1) \int ds_2 v(s_2) \exp\left(\mathrm{i}p \int_{s_1}^{s_2} dtv(t) \right) + \dots$$

  we integrate (Gaussian) over velocity and we get standard Feynman graphs in momentum space.

- **Ordinary quark loop in momentum space**

$$\langle \bar{q}q\bar{q}q \rangle \propto \int d^4k/(k^2(k+q)^2 + \dots \quad (4)$$

Conclusions for $W[C]$

- **Brownian paths with infinite cusps:**
  The velocity distribution being Gaussian, these paths are Brownian, with infinitely many cusps.
- **Path integral leads to new UV divergences.**
  The Wilson loop $W[C]$ for piecewise smooth loop is not the relevant physical observable here; insisting on making it finite contour-by-contour is not the right renormalization problem for current correlators.
- **The Wilson loop, integrated over paths** given the set of external momenta injected into the loop, creates observable momentum amplitudes which are renormalizable (just anomalous dimensions of quark currents).
- The renormalization factors of the quark currents are path independent, so they have nothing to do with the cusp singularities of $W[C]$ at the wedge where these currents act.

Parallel Transport and Loop Calculus

- **Holonomy Identity**. The path-ordered exponential of the covariant derivative operator $D_\mu(x)$ along the closed loop $C(\theta)$ with $C(\theta_0) = x$ is equal to the Wilson loop (holonomy) along the path $C(\theta)$ multiplied by the identity operator $\mathbb{I}$ in the Hilbert space.

$$\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x)\right) = \mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) A_\mu(C(\theta))\right) \otimes \mathbb{I} \tag{5}$$

- **Product integral** The path-ordered exponential on the L.H.S. is formally defined as the limit of a product integral:

$$\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x)\right) = \lim_{N\to\infty} \prod_{k=N\to 1} \exp\left(\Delta\theta_k \dot{C}_\mu(\theta_k) D_\mu(x)\right) \tag{6}$$

where $\Delta\theta_k \to 0$ and the product is ordered from right to left. Let $dC_\mu(\theta) = d\theta \dot{C}_\mu(\theta)$.

Parallel Transport and Loop Calculus

- **The Infinitesimal Disentangling** We analyze a single infinitesimal factor $\exp\left(dC_\mu D_\mu(x)\right)$. Using the Lie-Trotter product formula, $e^{A+B} = e^A e^B + O([A, B])$, we can split the operator:

$$\exp\left(dC_\mu(\partial_\mu + A_\mu(x))\right) = \exp\left(dC_\mu \partial_\mu\right) \exp\left(dC_\mu A_\mu(x)\right) + O(d\theta^2) \tag{7}$$

  The $O(d\theta^2)$ terms vanish in the $N \to \infty$ limit. The $\exp\left(dC_\mu \partial_\mu\right)$ operator is an infinitesimal translation operator, T_{dC}.

- **The Translation Identity** We now use the fundamental operator identity that defines a translation:

$$T_{dC}\, f(x) = f(x + dC)\, T_{dC} \tag{8}$$

Parallel Transport and Loop Calculus

- **The Hopping Identity** Applying this to our $A_\mu(x)$ operator, we find the "hopping" identity:

$$\exp\left(dC_\mu \partial_\mu\right)\exp\left(dC_\mu A_\mu(x)\right) = \exp\left(dC_\mu A_\mu(x + dC)\right)\exp\left(dC_\mu \partial_\mu\right) \tag{9}$$

  This "disentangling" allows us to move the (Abelian) derivative operator to the right.

- **Iteration and Re-ordering** We now apply this identity iteratively to the full product. Let $C_k = C(\theta_k) = x + \int_0^{\theta_k} d\theta' \dot{C}(\theta')$. By periodicity, $C_N = x$.

$$\prod_{k=N\to 1} \exp\left(dC_k \cdot D(x)\right) \approx \prod_{k=N\to 1} \left(\exp\left(dC_k \cdot A(C_k)\right)\exp\left(dC_k \cdot \partial_x\right)\right) \tag{10}$$

$$= \left(e^{dC_N \cdot A(C_N)} e^{dC_N \cdot \partial_x}\right) \dots \left(e^{dC_1 \cdot A(C_1)} e^{dC_1 \cdot \partial_x}\right) \tag{11}$$

Parallel Transport and Loop Calculus

- **operators are non-Abelian and remain path-ordered** The crucial point is that all ∂_x operators are Abelian and commute with each other, while all $A(C_k)$ operators are non-Abelian and remain path-ordered.
- **Arguments of gauge field shifted along the path** Furthermore, the resulting argument x_k of each A in the product at k-th place will be

$$x_k = C_N + \sum_{n=N\to k} dC_n = C_k \tag{12}$$

- **two distinct, ordered parts** This process separates the product into two distinct, ordered parts:

$$= \left(\prod_{k=N\to 1} e^{dC_k \cdot A(C_k)} \right) \left(\prod_{k=N\to 1} e^{dC_k \cdot \partial_x} \right) \tag{13}$$

Parallel Transport and Loop Calculus

- **Local limit** $N \to \infty$ Taking the $N \to \infty$ limit, this becomes:

$$= \left(\mathbb{P}\exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) A_\mu(C(\theta))\right)\right)\left(\exp\left(\int_0^{2\pi} dC_\mu \partial_\mu\right)\right) \tag{14}$$

- **The Closed Loop Condition** The first term is the Wilson loop $W[C]$. The second term is the identity operator because the loop C is closed:

$$\int_0^{2\pi} dC_\mu = \int_0^{2\pi} d\theta \dot{C}_\mu(\theta) = C_\mu(2\pi) - C_\mu(0) = 0 \tag{15}$$

- **Final form** The Hilbert space operator on the L.H.S. is equal to the Wilson loop multiplied by the identity operator $\mathbb{I}$.

Conclusions for $W[C]$

- **Base-point identity (unusual but correct):** The loop holonomy can be viewed as parallel transport of $D_\mu(x)$ *based at* $x = C(\theta_0)$. The gauge field is evaluated at the shifted points $C(\theta)$ after disentangling, even though the operator argument stays at x.
- **It is gauge invariant for the closed loop.** With the open loop, this parallel transport of covariant derivative operator is not gauge invariant: it transforms as bilocal operator

$$U(1,2) = T\exp\left(\int_1^2 d\theta C'_\alpha(\theta)\nabla_\alpha(x_1)\right); \tag{16}$$

$$U(1,2) \Rightarrow S(1)U(1,2)S^{-1}(2) \tag{17}$$

- **It becomes gauge invariant at closed loop**

$$\operatorname{tr} U(1,1) = W[C]\exp\left(\oint dsC'_\mu(s)\partial_\mu(x_1)\right) = W[C]\mathbb{I} \tag{18}$$

The Loop Space Derivative

To formulate dynamics, we treat the loop C as the fundamental degree of freedom.

- **The dot derivative :** D_μ.

$$\frac{\delta W}{\delta \dot{C}_\mu(\theta_0)} = \operatorname{tr} \mathbb{P} D_\mu(x) \exp\left(\int_0^{2\pi} d\theta \dot{C}_\mu(\theta) D_\mu(x)\right) \tag{19}$$

- **The Area Derivative:** $[D_\mu, D_\nu]$ Defined geometrically as the anti-symmetric tensor part of the second variation:

$$\frac{\delta W}{\delta \sigma_{\mu\nu}(\theta)} = \frac{\delta^2 W}{\delta \dot{C}_\mu(\theta - 0)\delta \dot{C}_\nu(\theta + 0)} - (\mu \leftrightarrow \nu) \tag{20}$$

- **The Loop Diffusion Operator:** $[D_\mu, [D_\mu, D_\nu]]$

$$L = \oint d\theta \dot{C}_\nu(\theta) \left(\frac{\delta}{\delta \dot{C}_\mu(\theta + 0)} - \frac{\delta}{\delta \dot{C}_\mu(\theta - 0)}\right) \frac{\delta}{\delta \sigma_{\mu\nu}(\theta)} \tag{21}$$

Leibniz property

- $L(A[C]B[C]) = L(A)B + AL(B)$
- $Lf(A[C]) = L(A[C])f'(A[C])$
- Note: This allows for an ansatz $W[C] = W_{pert}[C]\, e^{-\kappa Area[C]}$.

The Makeenko–Migdal (MM) Equation

- The equation of motion for the Wilson loop ($N_c \to \infty$):

$$\boxed{L(W[C]) = \lambda \oint_C dx_\mu \oint_C dy_\mu \delta^{(4)}(x-y) W[C_{xy}] W[C_{yx}]} \tag{22}$$

- **Left Side:** The Loop Diffusion operator.
- **Right Side:** The interaction term (loop splitting).
- **Factorization:** $W_2(C_1, C_2) \to W(C_1)W(C_2) + O(1/N_c^2)$.

Graphic form of MM equation

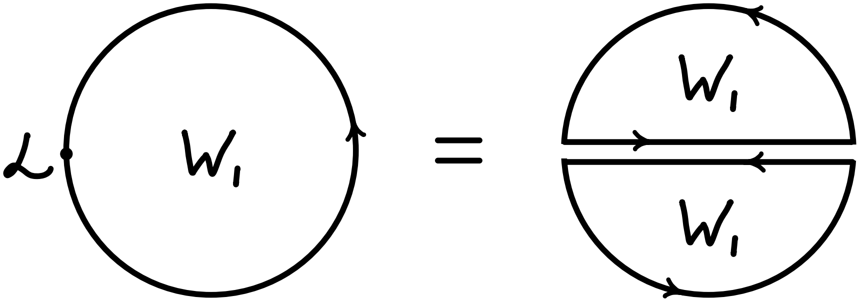


Figure: The MM equation with the loop diffusion operator and the integral term.

Generating Planar Graphs (The Bootstrap)

- Starting with $W = 1$, iterations generate the "Frame Diagrams."
- The inversion of ∂^2 generates the gluon propagator $1/(x-y)^2$.
- The commutators $[\partial, \partial]$ generate the correct tensor structures.
- **Result:** The equation sums all planar Feynman diagrams.

Discussion map: choose a fork

We can go in (at least) three directions. Please interrupt and pick:

Fork A: Planarity / combinatorics

- why RHS is a *product*
- cyclic order $\Rightarrow$ planar recursion
- how iterations reproduce planar graphs

Fork B: UV/contact + regularization

- $\delta(x - y)$ at self-intersection
- cusps / Brownian loops
- analytic point-splitting ("wires")

Fork C: Momentum loop space

- finite algebraic form
- off-shell recursion viewpoint
- continuum limit after solving

(We can take one fork deeply, then return and take another.)

Fork A: how the loop equation generates planar graphs

- Large-N input: *factorization* of multi-loop correlators turns the loop equation into a nonlinear recursion.
- The $\delta(x-y)$ forces two points on the *same oriented loop* to coincide in spacetime; this induces a canonical split $C = C_{xy} \cdot C_{yx}$ respecting cyclic order.
- The RHS becomes $W[C_{xy}]\, W[C_{yx}]$: "split one boundary into two boundaries" with preserved ordering.
- That preserved boundary order is the combinatorial hallmark of planarity (double-line viewpoint).
- Iterating from $W = 1$ builds planar "frame diagrams"; windows are then "glassed" by Wilson-loop factors.

Fork B: why coordinate space is singular, and what regularization does

- The loop equation's interaction term is a *contact* term: it lives at self-intersections ($x = y$).
- Dominant loops in the path integral are irregular (cusps / Brownian-like), so the contact term becomes ill-defined.
- Analytic regularization: soften the UV singularity without changing $d = 4$ (heuristically: $1/k^2 \to 1/(k^2)^{1+\varepsilon}$).
- Implemented as gauge-invariant point splitting: replace $\delta(x - y)$ by a smeared kernel and close split loops by short path factors ("wires").
- Purpose: make the loop equation well-defined *before* discussing continuum limits or nonperturbative solutions.

Fork C: momentum loop space (why amplitudes people should care)

- Coordinate-space pathologies are UV/contact issues (cusps, $\delta(x-y)$), not a failure of planarity.
- In momentum loop space the delta-contact catastrophe disappears and the loop equation becomes a finite local functional differential equation in $P(\theta)$.
- The recursion is naturally *off-shell*: splitting/merging operations act on loop-space kinematics, analogous in spirit to planar recursion for integrands.
- Practical message: solve dynamics first in the finite momentum-loop formulation, then take limits (e.g. remove smearing) after the solution is controlled.

Phys. Rep. '83 excerpt: what to look for (1–2 minutes)

- Next I'll show a few *verbatim pages* from the Phys. Rep. section *"Analytic regularization of the loop equation"* (no re-derivation on the slides).
- Purpose: make precise how the singular contact term in the MM equation is rendered well-defined (before taking any continuum limits).
- Watch for three ingredients:
  - **Analytic softening** of the UV singularity (a controlled ε-deformation).
  - **Gauge-invariant point splitting:** replace $\delta(x-y)$ by a smeared kernel.
  - **"Wires" / gluon path closure:** close the split loops with short path factors to preserve gauge invariance.
- After these pages, I will return to the compact "bootstrap" form (eqs. (18)–(23) in these slides) and explain how iteration reproduces planar graphs.

From Phys Rep '83 (1)

Analytic regularization of the loop equation

The most convenient regularization for perturbative gauge theory is the famous dimensional regularization, which was in fact implied above. However, it turns out to be ill-defined beyond perturbation theory. For example, it misses such an important phenomenon as the axial anomaly, which requires explicit point splitting.

The version of the point splitting procedure closest to dimensional regularization is analytic regularization. Now we are going to adapt analytic regularization for the loop equation. The idea is to replace the propagator $k^{-2} \to k^{-2-2\varepsilon}$ in a gauge invariant manner. The regularization will consist in analytic continuation in ε from the domain of convergence. The number of dimensions $d = 4$ will be unchanged, so the loop functionals will be well defined. Only the coefficient functions, i.e. the traces of gluon Green functions, will become nonsingular at coinciding points.

With the gluon propapator $k^{-2-2\varepsilon}$ in planar graphs the above line with a cross in (3.51) will correspond to $k^{-2\varepsilon}$ since the vertex (3.47) will remain $\sim k^2$ as before. At finite ε the points x and y would be split with the weight $\varepsilon|x-y|^{2\varepsilon-4}$ instead of the δ-function.

Naturally there should arise extra graphs to compensate the violation of gauge invariance at the point splitting. In the old formulation it corresponds to path factors $U(\Gamma_{xy})$ in the adjoint representation where Γ_{xy} is some path between x and y. In the loop equation the paths Γ_{xy}, Γ_{yx} should be added to close the loops

From Phys Rep '83 (2)

$$W(C_{xy}) \to W(C_{xy}\Gamma_{xy}^{-1}) \quad (3.54)$$

$$W(C_{yx}) \to W(C_{yx}\Gamma_{xy})\,. \quad (3.55)$$

From the point of view of gauge invariance these paths are arbitrary but space symmetry would be violated for arbitrary paths. There is a natural choice of these paths preserving the symmetry. Namely they may be chosen to coincide with the path of the gluon. The extra vertices arising from the factor $(U(\Gamma_{xy}))^{\text{adj}}$ can now be interpreted as gluon interactions. Roughly speaking this is the same as replacing the δ-function by a covariant derivative in the adjoint representation $\nabla_\mu = \partial_\mu + [A_\mu,$ raised to the nonintegral power $(-\nabla_\mu^2)^{-\epsilon}$.

Let us now make these ideas more precise. First of all it is convenient to consider the integrated version of the loop equation (with $L = \int_C \mathrm{d}x_\nu L_\nu$)

$$L\, W(C) = \lambda \int_C \mathrm{d}x_\mu \int_{C_{xx}} \mathrm{d}y_\mu\, \delta^d(x-y)\, W(C_{xy})\, W(C_{yx})\,. \quad (3.56)$$

From Phys Rep '83 (3)

The original local version represented a vector equation for the scalar functional $W(C)$, so it was overcomplete. There were certain consistency relations following from the identities (2.88). The r.h. side satisfied the selfconsistency relations due to conservation of the vector current $j_\nu(C_{xx})$ in the sense of the loop calculus $\partial^\nu(x)\, j_\nu(C_{xx}) = 0$. Point splitting in the original equation should respect selfconsistency conditions.

There is no real problem, since there are no anomalies in the vector current but it leads to unnecessary complications. The scalar equation (3.56) is, in principle, equivalent to the overcomplete system, but it is more convenient for point splitting.

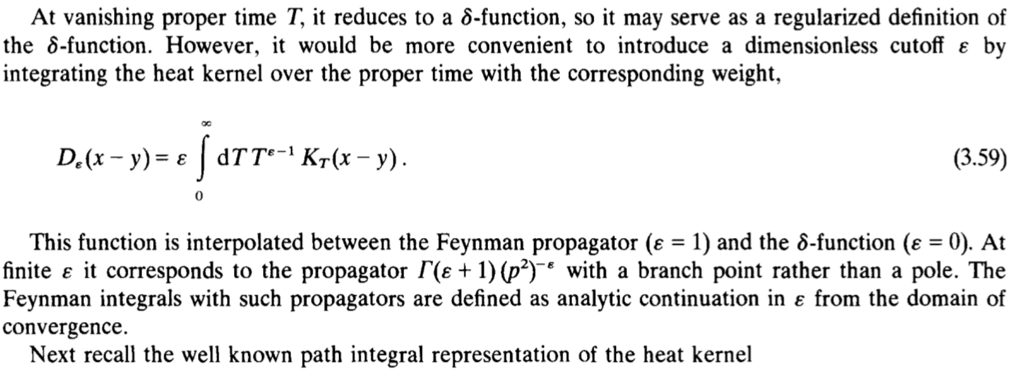
As a first step in the gauge invariant point splitting procedure let us introduce the heat propagation kernel

$$K_T(x - y) = \langle x|\, \exp(-T\hat{p}^2)|y\rangle \tag{3.57}$$

$$\hat{p}_\mu = -\mathrm{i}\; \partial/\partial x_\mu\,. \tag{3.58}$$

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

From Phys Rep '83 (4)



At vanishing proper time T, it reduces to a δ-function, so it may serve as a regularized definition of the δ-function. However, it would be more convenient to introduce a dimensionless cutoff ε by integrating the heat kernel over the proper time with the corresponding weight,

$$D_\varepsilon(x-y) = \varepsilon \int_0^\infty \mathrm{d}T\, T^{\varepsilon-1}\, K_T(x-y)\,. \tag{3.59}$$

This function is interpolated between the Feynman propagator ($\varepsilon = 1$) and the δ-function ($\varepsilon = 0$). At finite ε it corresponds to the propagator $\Gamma(\varepsilon+1)\,(p^2)^{-\varepsilon}$ with a branch point rather than a pole. The Feynman integrals with such propagators are defined as analytic continuation in ε from the domain of convergence.

Next recall the well known path integral representation of the heat kernel

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

From Phys Rep '83 (5)

$$K_T(x-y) = \int_x^y Dx(\cdot)\exp\left(-\int_0^T dt\,\tfrac{1}{4}\dot{x}^2\right). \tag{3.60}$$

We shall use a special notation for such a path integral (including the proper time integration)

$$\int_0^\infty dT \int_x^y Dx(\cdot)\exp\left(-\int_0^T dt\,\frac{\dot{x}^2}{4}\right)\cdots \equiv \int D\Gamma_{xy}\cdots . \tag{3.61}$$

With this notation

$$D_\varepsilon(x-y) = \int D\Gamma_{xy}\,\varepsilon T^{\varepsilon-1}. \tag{3.62}$$

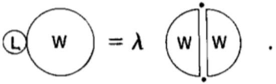
Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

From Phys Rep '83 (6)

The reader is warned that sometimes other definitions of the path integral are used in the literature, in particular in the author's papers. We use this one here, since it is most traditional and (therefore) the easiest to memorize. Now we are ready to write down the regularized loop equation

$$L\, W(C) = \varepsilon\lambda \int_C \mathrm{d}x_\mu \int_C \mathrm{d}y_\mu \int \mathrm{D}\Gamma_{xy}\, T^{\varepsilon-1}\, W(C_{xy}\,\Gamma_{yx})\, W(C_{yx}\,\Gamma_{xy})\,. \tag{3.63}$$

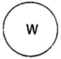
The iteration of this equation would produce regularized perturbation theory. By construction the first iteration yields the one gluon graph with propagator $\Gamma(\varepsilon+1)\,(k^2)^{-\varepsilon-1}$. In higher orders extra vertices will arise. The following graphical representation of (3.63) proves to be convenient:



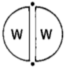
(3.64)

Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

Here the functional integration over paths Γ_{xy} corresponding to the double line and the integration over the end points x, y is implied. Each window corresponds to the W-factor of the corresponding loop. We refer to such a graph as a “glassed” graph. Note that only planar graphs can be glassed. We have already dealt with glassed graphs in the random matrix model, in which they were only a technical device. Here the graphs have to be understood in the usual Feynman sense of summing the corresponding amplitudes over all histories. The iterations of this equation can be performed in an integral form



$$= 1 + \lambda L^{-1}$$



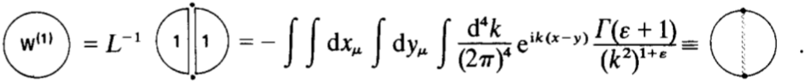
. (3.65)

In the first order on the right-hand side,

$$= \int_C dx_\mu \int_C dy_\mu \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^{2\varepsilon}} \Gamma(\varepsilon + 1)\,. \tag{3.66}$$

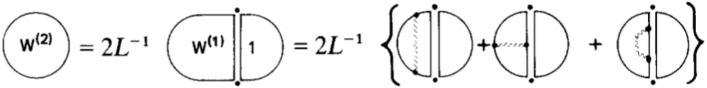
Figure: Analytic regularization of the loop equation (Phys. Rep. 1983)

The application of the inverse L operator (with the Euclidean boundary conditions) reduces to multiplication by $-k^{-2}$ in momentum space, i.e.



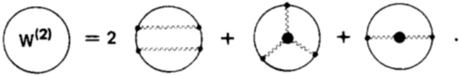
(3.67)

In the next order we find



(3.68)

where the wavy line denotes the regularized gluon propagator $\delta_{\mu\nu}\Gamma(\varepsilon+1)/k^{2+2\varepsilon}$. Application of L^{-1} is here a bit more tedious, but straightforward. The result has the structure



(3.69)

Bootstrap equation in detail

- $$\frac{\delta W[C]}{\delta \sigma_{\mu\nu}(x)} = \left(\partial^x_\nu \delta_{\mu\alpha} - \partial^x_\mu \delta_{\nu\alpha}\right) B_\alpha[C_{xx}] + B_{\mu\nu}[C_{xx}]; \tag{23}$$

- The linear terms related to the source term $J_\nu[C_{xx}]$:

$$B_\nu = \left(-\partial^2 \delta + [\partial, \partial]\right)^{-1} \left(J_\lambda - \partial_\mu B_{\mu\lambda}\right); \tag{24}$$

$$J_\nu[C_{xx}] = \lambda \int dy_\nu \delta^4(x-y) W[C_{xy}] W[C_{yx}] \tag{25}$$

- Expansion of the inversion operator:

$$\left(-\nabla^2 + [\nabla, \nabla]\right)^{-1} = -\nabla^{-2} \\ - \nabla^{-2}[\nabla, \nabla]\nabla^{-2} - \nabla^{-2}[\nabla, \nabla]\nabla^{-2}[\nabla, \nabla]\nabla^{-2} + \dots \tag{26}$$

- We use the identity:

$$[\nabla_\alpha, \nabla_\beta] F[C_{xx}] = \frac{\delta F[C_{xx}]}{\delta \sigma_{\alpha_\beta}(y)} \Big|_{y=x+0}^{y=x-0} \tag{27}$$

- Representation as a sum over the Brownian path Γ:

$$- \nabla^{-2} J_\nu[C_{xx}] = \lambda \int d^4 z \int \mathcal{D}\Gamma_{xz}$$
$$\oint_{\Gamma_{zx} \cdot C_{xy} \cdot C_{yx} \cdot \Gamma_{xz}} dy_\nu \delta^4(z-y) W[\Gamma_{zx} \cdot C_{xy}] W[C_{yx} \cdot \Gamma_{xz}] \tag{28}$$

Graphic Form of Bootstrap equation

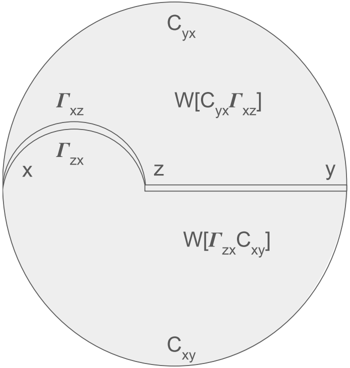


Figure: The path integral in (28) with Brownian paths.

Gluon graphs by iterations of Bootstrap equation

- In the old MM papers, the bootstrap equation was iterated in λ, starting with $W[0] = 1$.
- It faithfully reproduced conventional planar graphs, including ghost loops.
- One recovers the (planar) perturbative beta-function in this bootstrap iteration.

Summary of the Bootstrap Approach 1

- The planar MM equation (22) transforms into a path integral equation.
- Frame diagrams look like planar trees with windows glassed with W functionals.
- Sums over Brownian paths produce gluon propagators.

Summary of the Bootstrap Approach 2

- The frame diagram is manifestly gauge invariant.
- Iterative solutions recover uniquely Faddeev-Popov perturbation theory.
- Various parts of the loop integral correspond to graphs including ghost loops.

The Coordinate Space Catastrophe

1. **The Delta Function:** $\delta^{(4)}(x-y)$ forces interaction at a single point.
2. **The Cusp Singularity:**
   - Physical quark paths are nowhere differentiable.
   - Perturbative QCD dictates a divergence:

$$\log W \sim -\Gamma_{cusp}(\gamma)\log(\Lambda_{UV}L) \tag{29}$$

3. **Natural question:** do cusp/perimeter divergences of $W[C]$ imply the loop equation itself must be renormalized?

Self - intersection and four wedges

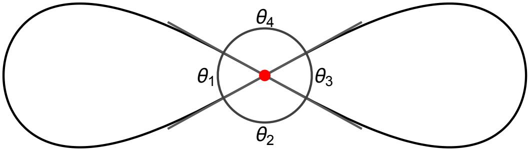


Figure: Four wedges at self intersection

Why multiplicative cusp renormalization is not closed at intersections (1)

- Consider a self-intersecting contour $C = C_{xy} \cdot C_{yx}$ with intersection at $x = y$.
- Geometrically, there are **four** local wedge angles $\theta_1, \ldots, \theta_4$ at the crossing.
- Cusp renormalization is angle-dependent: $W[C] \to \prod Z(\theta_i)\, W_{\text{fin}}$.
- **The Mismatch:**
  - **LHS:** In the MM equation the LHS is $L(W[C])$:

$$\text{LHS} \sim Z(\theta_1)Z(\theta_2)Z(\theta_3)Z(\theta_4)L(W_{\text{fin}}) + W_{\text{fin}}L(Z(\theta_1)Z(\theta_2)Z(\theta_3)Z(\theta_4))$$

  - **RHS:** The equation factorizes: $W[C_{xy}]W[C_{yx}]$. Each W sees only its own single cusp:

$$\text{RHS} \sim \underbrace{Z(\theta_1)}_{\text{from } W(C_{xy})} \cdot \underbrace{Z(\theta_3)}_{\text{from } W(C_{yx})} \lambda \int W_{\text{fin}}(C_{xy})W_{\text{fin}}(C_{yx})\delta(x-y)$$

- The missing factors $Z(\theta_2)Z(\theta_4)$cannot be absorbed into the bare coupling λ, because λ is universal and cannot depend on local angles θ_2, θ_4. These factors must cancel themselves on the left side.

How the loop diffusion operator cancels divergent diagrams

- The operator L cancels divergent factors on the LHS, related to the θ_2, θ_4 wedges.
- This operator, when applied to the planar graphs in $W[C]$, generates extra terms, exactly canceling all the graphs with gluon lines connecting C_{xy} with C_{yx}.
- The remaining gluon line connects points x, y on the loop, and this line, after the application of ∂_x^2 in L, becomes $\delta(x-y)$ matching the RHS. The other lines crossing this one, all get cancelled.
- The renormalization factors $Z(\theta_2), Z(\theta_4)$ depend on $\theta_{2,4} = \arccos\left((v(x) \cdot v(y))/|v(x)||v(y)|\right)$. So, one can apply the functional derivative $\frac{\delta}{\delta v(x)}, \frac{\delta}{\delta v(y)}$. Results are singular, so one has to keep regularization finite.
- So, the renormalization of $W[C]$ by cusp factors does not lead to a finite renormalized loop equation: operator L does not have a local limit.
- **Conclusion:** A single closed coordinate-space equation for $W[C]$ is *not stable* under renormalization. One has to keep regularization finite in that equation, and only go to local limit later, in observables, such as amplitudes for quark bilinears.

RG: local operators vs. Wilson loops on Brownian paths

- **QCD renormalization (RG) is formulated for local gauge-invariant operators:**

$$\mathcal{O}_i^{\text{bare}} = Z_{ij}(\mu)\,\mathcal{O}_j(\mu), \qquad \mu\frac{d}{d\mu}\mathcal{O}_i(\mu) = \gamma_{ij}(g(\mu))\,\mathcal{O}_j(\mu).$$

- **Wilson loops are nonlocal line functionals.** Their UV structure depends on local contour geometry (perimeter/cusps/intersections) rather than on a local operator basis .
- **In physical correlators the loop variable is integrated over a Brownian ensemble:** typical quark paths are nowhere differentiable $\Rightarrow$ infinite cusp density . Therefore a "cusp counterterm per cusp (or per wedge)" cannot define a sensible RG notion for $W[C]$ on the paths that actually dominate the observable path integral .
- **Conclusion / strategy:** do not demand a renormalized coordinate-space $W[C]$ contour-by-contour; instead regularize geometrically and pass to **momentum loop space** where the loop equation is finite (no contact $\delta(x-y)$ singularity) and the local limit is taken *after* solving .

Summary of Lecture I

1. **The MM Loop Equation** provides the defining planar loop dynamics (Schwinger–Dyson form) for Wilson loops in large-N_c QCD.
2. **The observable amplitudes** are given by singular path integrals over Brownian loops involving $W[C]$ and Dirac amplitudes for the free Dirac particle propagation around the loop.
3. **Coordinate Space:** Fixed-contour $W[C]$ has cusp/perimeter UV singularities; in the physical Brownian-loop path integral these are not the renormalization problem that controls observable current correlators.
4. **The Way Forward:** Regularization by geometry and momentum space, where there are no singularities in the loop equation.

Next Lecture:

Momentum Loop Dynamics and the Finite Algebraic Structure of QCD.