Bootstrapping Large $N$ confining gauge theories

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w/ Leonardo Rastelli
Bootstrap philosophy

UV: Confining gauge theory. \( \{ m_i^2, \lambda_{ijk} \} \)
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**IR:** EFT for lowest excitations \( \phi \).

\[
\mathcal{L}_{\text{EFT}} = -\frac{1}{2} (\partial \phi)^2 + g_0 \phi^4 + g_1 \phi^2 (\partial \phi)^2 + g_2 (\partial \phi)^4 + \cdots
\]
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\]

**Target:** Large \( N \) QCD.

\[
U(N_f)_L \times U(N_f)_R \rightarrow U(N_f)_V
\]

\( N_f^2 \) GB = \( \pi^a \)
Exclusion plot

\[ \tilde{g}_2 \equiv \frac{g_2 M^2}{g_1} \]

\[ \tilde{g}'_2 \equiv \frac{g'_2 M^2}{g_1} \]

Large $N$ QCD?
Beyond pions

External rho mesons:
Beyond pions

External rho mesons:

Probe photons:

Access the anomaly
Correlation Functions of Local Operators in the $TT\bar{T}$-Theory from Topological Gravity

Presented by: Netanel Barel
Supervised by: Ofer Aharony

Weizmann Institute of Science

July 13, 2023
Overview

1. Introduction
2. Definitions
3. JT formalism
The first sentence of Leo Tolstoy’s novel Anna Karenina is: “Happy families are all alike; every unhappy family is unhappy in its own way.”
Introduction

- The first sentence of Leo Tolstoy’s novel Anna Karenina is: ”Happy families are all alike; every unhappy family is unhappy in its own way.”
- The first sentence of a QFT course should be: ”Local QFTs are all alike; every non local QFT is non local in its own way.”
Introduction

- RG flow of Local QFTs: CFT in the \textbf{UV} to CFT in the \textbf{IR}. 

Introduction

- RG flow of Local QFTs: CFT in the UV to CFT in the IR.
- $T\bar{T}$ deformation: CFT in the IR and non-local QFT in the UV.
Definitions

- Theory in 1+1d with energy-momentum tensor $T_{\alpha \beta}$, we define the composite operator:

**Definition 1 - $T^\dagger$ Operator**

$$T^\dagger \equiv \det (T_{\alpha \beta})$$
Definitions

- We modify the Lagrangian:

\[ \mathcal{L}^{(t+dt)} = \mathcal{L}^{(t)} + T\bar{T}^{(t)} dt, \]

\( T\bar{T}^{(t)} \) is energy-momentum tensor determinant of \( \mathcal{L}^{(t)} \).
Previous results

- Leading order correction:

\[ C^t(q) = |q|^{2\Delta} \left( 1 + \frac{tq^2}{\pi} \ln \left( \frac{|q|}{\mu} \right) \right) \]
Previous results

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- For the high momentum limit Cardy calculated:

\[ C^t(q) = |q|^{2\Delta} \left( \frac{|q|}{\mu} \right)^{\frac{tq^2}{\pi}} \]
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- In a CFT the correlator looks like: \[ |q|^{2\alpha} \]
JT formalism

- The undeformed theory coupled to JT gravity:

**Definition 3 - JT Action**

\[ S_{\text{TT}}(\psi, e^a_\alpha, X^a) = S_0(\psi, g_{\alpha\beta}) + S_{JT}(X^a, e^a_\alpha) \]

\[ S_{JT}(X^a, e^a_\alpha) \equiv -\frac{1}{2t} \int d^2\sigma \epsilon^{\alpha\beta} \epsilon_{ab}(\partial_\alpha X^a - e^a_\alpha)(\partial_\beta X^b - e^b_\beta). \]
JT formalism

- The undeformed theory coupled to JT gravity:

**Definition 3 - JT Action**

\[
S_{TTT}(\psi, e^a_\alpha, X^a) = S_0(\psi, g_{\alpha\beta}) + S_{JT}(X^a, e^a_\alpha)
\]

\[
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\]

- Non-perturbative formulation of the TTT deformation.
JT formalism

- local operators - deformations of an undeformed operators $O(\sigma)$ in position and momentum space are:

**Definition 4 - Deformed Operator**

\[
O(X_0) = \int d\sigma \sqrt{g(\sigma)}O(\sigma)\delta(X(\sigma) - X_0)),
\]

\[
O(q_0) = \int d\sigma \sqrt{g(\sigma)}O(\sigma)\exp(iq_0X(\sigma)).
\]
The correlation function in momentum space:

\[
\langle \mathcal{O}(q_1)\mathcal{O}(q_2) \rangle \equiv \frac{1}{Z_{TT}} \int \frac{D\mathbf{e}D\mathbf{x}D\psi}{V_{\text{diff}}} \mathcal{O}(q_1)\mathcal{O}(q_2)e^{-S_{TT}}
\]

\[
Z_{TT} \equiv \int \frac{D\mathbf{e}D\mathbf{x}D\psi}{V_{\text{diff}}} e^{-S_{TT}}.
\]
Main Result

- High momentum limit (non-perturbative):

\[ |q|^{2\Delta} \left( \frac{|q|}{\mu} \right)^{-\frac{t|q|^2}{\pi}} \]

- Remind high momentum limit of Cardy (perturbative):

\[ |q|^{2\Delta} \left( \frac{|q|}{\mu} \right)^{\frac{tq^2}{\pi}} \]
Generalized Symmetries and Noether’s theorem in QFT

Valentin Benedetti

Based on work with Horacio Casini and Javier Magan

arXiv: 2205.03412, 2212.11291
When is there a well-defined Noether current for a continuous symmetry in QFT?

1. A priori **Noether's theorem** asserts the existence of a local conserved current when an action is invariant under a continuous symmetry group. However, a long-standing question is to determine to what extent, or under what conditions, this theorem holds in QFT.

2. Known counter-examples (the Noether current is not gauge invariant)
   - Duality Symmetry in Free Maxwell theory
   - **Weinberg Witten theorem** (for example, graviton stress-tensor)
   - Chiral symmetry and the ABJ anomaly
   - etc

3. What is peculiar about the symmetry in these cases?

4. What does this teach us about the completeness of the spectrum?
Given a theory with generalized symmetries and a global (0-form) symmetry described by a continuous symmetry group $G$:

If the **generalized symmetries are charged** under the action of the continuous symmetry group $G$

i.e. There are non-local operators that transform under the action of the group $G$.

$\Rightarrow$

There is no **Noether current** for $G$

However, the symmetry is still implemented locally by twist operators

(Weak version of Noether's theorem)

Rederives Weinberg-Witten theorem

e.g. Generalized Symmetries of the free graviton are charged under space-time symmetries ($\Rightarrow$ no stress-tensor).

Same applies to all free massless particles of spin $\geq 3/2$.

$\Leftarrow$?

Non-local operators produce Haag duality violations, then always come in dual classes assigned to complementary regions.

In this case, they both must form a continuum

Haag duality defects form a **continuous non-compact group**

$\Rightarrow$

QFT with a **free massless sector** or the theory needs to be UV completed with charges that break all generalized symmetries

$\Rightarrow$

Charges in any UV completion of:

- neutral and non-linear electrodynamics
- interacting goldstone modes

Haag duality defects form a **continuous compact group**

e.g. **ABJ anomaly** and chiral symmetry
THANK YOU!
A brief review of Coon amplitudes

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Princeton Institute for Advanced Study
PiTP 2023: Understanding Confinement
July 13, 2023
Coon amplitudes: old but back in style

[Coon 1969] [https://inspirehep.net/literature/56699] [slide inspired by Remmen 2023]
Coon amplitudes: long history & open problems

- 1968: Veneziano discovery
  
- 1969: Coon discovery
  
- 1970s: \(N\)-pt, operator formulation, pheno
  
- 1988: independent rediscovery
  
- 2016: S-matrix bootstrap counter-ex
  
- 2022–: physical realization? (open strings on D-branes in AdS)
  
- 2022–: generalizations?
  
- 2022–: unitarity?
Coon amplitudes: family of ampl’s parametrized by $q \geq 0$

- 4-pt, $q$-gamma func’s

$$\mathcal{A}^{(4)}_q(s,t) = q^{\alpha_q(s)\alpha_q(t)} \frac{\Gamma_q(-\alpha_q(s))\Gamma_q(-\alpha_q(t))}{\Gamma_q(-\alpha_q(s) - \alpha_q(t))} \xrightarrow{q \to 1} \mathcal{A}_{\text{Ven}}^{(4)}(s,t)$$

- Regge traj.

$$\alpha_q(s) = \log_q[1 + (q - 1)(s/\mu^2 - \delta)]$$

- spectrum

$$m^2_n = \mu^2\left(\frac{1-q^n}{1-q} + \delta\right)$$

- $q = 0, \infty \Rightarrow$ scalars
- $q = 1 \Rightarrow$ strings (super $\delta = 0$, bosonic $\delta = -1$)
- $q < 1 \Rightarrow$ accumulation pt ($m^2_\infty = \mu^2\left(\frac{1}{1-q} + \delta\right)$) & non-meromorphic
- $q \geq 1 \Rightarrow$ unbounded ($m^2_\infty = \infty$) & meromorphic

- unitarity $\Rightarrow q \leq 1 \& -1 \leq \delta \leq \frac{1}{3}$

[Figueroa, Tourkine 2022]
Coon amplitudes: problems at higher points

- 5-pt, $q$-hypergeometric func's

$$A_q^{(5)} = A_q^{(4)}(s_{12}, s_{23}) A_q^{(4)}(s_{34}, s_{45})$$

$$\times 3\Phi_2\left[ q^{\alpha_q(s_{12})+\alpha_q(s_{23})+\alpha_q(s_{34})+\alpha_q(s_{45})} q^{\alpha_q(s_{12})+\alpha_q(s_{23})-\alpha_q(s_{51})} q^{-1} q^{\alpha_q(s_{51})} \right]$$

- $q \leq 1$ & special kinematics $\implies A_q^{(4)}$ from [Cheung, Remmen 2023]

- $q \geq 1$ $\implies$ factorization in all channels

- $q < 1$ $\implies$ factorization in subset of channels

- $N$-pt generalization for $q > 1$ (non-unitary)
Thank you!

Questions?
On Unitarity of the Coon Amplitude

Rishabh Bhardwaj & Shounak De
(Brown University)

Based on:
• 2212.00764 (RB, SD, M. Spradlin, A. Volovich)
• 2208. xxxxx (RB, SD - to appear)
Veneziano amplitude (1968):

\[ A^\gamma(s,t) = \frac{\Gamma(-\alpha_0-s) \Gamma(-\alpha_0-t)}{\Gamma(-2\alpha_0-s-t)} \]

i) crossing symmetric in \( s \leftrightarrow t \)

ii) polynomial residues

iii) meromorphic

1969: Worldsheet realization as a dual resonance model \( \rightarrow \) 50+ years of ST!
**Question:** Is $A^v(s,t)$ unique?

**Coon amplitude (1969):**

$$A^c(s,t) = 2^{\alpha_2(s)\alpha_2(t) - \alpha_2(s) - \alpha_2(t)} \frac{\Gamma_2(-\alpha_2(s)) \Gamma_2(-\alpha_2(t))}{\Gamma_2(1 - \alpha_2(s) - \alpha_2(t))}$$

where $\alpha_2(s) = \frac{\ln(1-(1-2)(\alpha_0+s))}{\ln 2}$.

1) $\lim_{q \to 1} A^c(s,t) = A^v(s,t)$

2) $\lim_{q \to 0} A^c(s,t) = \frac{1}{s+\alpha_0} + \frac{1}{t+\alpha_0} + 1$
**Question:** Is \( A^c(s,t) \) unitary?

**Answer:** Yes, in some regimes!

\[ A^c(s,t) \rightarrow \frac{1}{s-[n]_2} \times R^c_{[n]_2}(t) \rightarrow \text{residue polynomials} \]

**Condition of unitarity:** \( R^c_{[n]_2}(t) = \sum_{j} B_{n;j}^{(0)}(z) g_{j}^{(0)}(t) \) 

with \( B_{n;j}^{(0)} \geq 0 \)

(22.12.0764) (RB, SDe, M.Spadlin, A.Votovich)
**Question:** What are the ST origins of $A^c(s,t)$?

**Answer:** Maybe none (wip - RB, SD)

$$\text{SL}(2,\mathbb{C}) \xrightarrow{q\text{-def.}} \text{SL}(2,\mathbb{C})_q \xrightarrow{\text{construct } q\text{-CFTs}}$$

$$q\text{-deformed amplitudes} \xrightarrow{\text{q-Ward identities}} q\text{-deformed } \langle \rangle$$

NO observables that resemble $A^c(s,t)$!
Quasinormal modes of the D0 brane black hole

Anna Biggs

Princeton University

July 2023
Near-horizon limit of $D3$ branes $\rightarrow AdS_5 \times S^5 = SYM$ in $4d$
Near-horizon limit of $D3$ branes $\rightarrow \text{AdS}_5 \times S^5 = \text{SYM in 4d}$

There is an analogous duality for $p \neq 3$, but the Yang-Mills theory is not conformally invariant. Itzhaki, Maldacena, Sonnenschein, Yankielowicz 1998
Dp brane holography

Near-horizon limit of D3 branes \( \rightarrow \) \( \text{AdS}_5 \times S^5 = \text{SYM in 4d} \)

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Supergravity geometry: conformal to \( \text{AdS}_{2+p} \times S^{8-p} \)

\[
\begin{align*}
\text{ds}_{\text{string}}^2 & \propto z^{\frac{3-p}{5-p}} \left( \frac{-dt^2 + dx_p^2 + dz^2}{z^2} + \mathcal{R}^2 d\Omega_{8-p}^2 \right), & \mathcal{R} & = \frac{5-p}{2} \\
e^{\phi} & \propto z^{\frac{(7-p)(3-p)}{2(5-p)}}, & A_0...p & \propto z^{-\frac{2(7-p)}{5-p}}
\end{align*}
\]
A black hole described by a quantum mechanics

today: $p = 0$ case

near-horizon geometry of charged BH = BFSS matrix model
(D0 brane supergravity solution) (low temp regime)
Scaling similarity

The $AdS_{2+p} \times S^{8-p}$ supergravity geometries

$$ds_{\text{string}}^2 \propto z^{\frac{3-p}{5-p}} \left( \frac{-dt^2 + dx^2_p + dz^2}{z^2} + \mathcal{R}^2 d\Omega_{8-p}^2 \right), \quad \mathcal{R} = \frac{5-p}{2}$$

$$e^\phi \propto z^{\frac{(7-p)(3-p)}{2(5-p)}}, \quad A_{0...p} \propto z^{\frac{2(7-p)}{5-p}}$$
Scaling similarity

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\[
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    ds^2_{\text{string}} &\propto z^{\frac{3-p}{5-p}} \left( \frac{-dt^2 + dx_p^2 + dz^2}{z^2} + \mathcal{R}^2 d\Omega_{8-p}^2 \right), & \mathcal{R} = \frac{5-p}{2} \\
    e^\phi &\propto z^{\frac{(7-p)(3-p)}{2(5-p)}}, & A_{0\ldots p} &\propto z^{\frac{2(7-p)}{5-p}}
\end{align*}
\]

have the property that, under a coordinate rescaling

\[
t \to \gamma t, \quad z \to \gamma z, \quad x \to \gamma x
\]

the action gets rescaled:

\[
S \to \gamma^{-\bar{\theta}} S, \quad \bar{\theta} = \frac{(3-p)^2}{5-p}
\]

Not a symmetry, but a “similarity” Landau, Lifschitz 1982
Perturbations of D0 black hole

How does the system respond to perturbations?

In the bulk: black hole quasinormal modes, $\omega_i \in \mathbb{C}$

In the quantum system: thermalization timescale

A simple observable in the matrix model we can predict using gravity.
Perturbations of D0 black hole

Finding wave equation for linearized fluctuations around D0 brane background = hard
Perturbations of D0 black hole

Finding wave equation for linearized fluctuations around D0 brane background = hard

Tricks:

1) View $Dp$ brane geometry as the dimensional reduction of $AdS_{2+p+\bar{p}} \times S^8$. Kantischeider, Skenderis 2009

2) Masses of perturbations can be classified using the scaling similarity.

(Computed previously by Sekino, Yoneya 2000)
Quasinormal mode spectrum

\[ \Delta = 28/5 \text{ QNM} \]

\[ \omega_n = 2\pi T \alpha_n \]
Entanglement entropy from non-equilibrium lattice simulations

Andrea Bulgarelli
Università degli Studi di Torino and Istituto Nazionale di Fisica Nucleare

Based on: A. Bulgarelli and M. Panero JHEP 06 (2023) 030
Entanglement in (L)QFT

\[ S(A) = - \text{Tr}(\rho_A \log \rho_A) \quad S_n(A) = \frac{1}{1 - n} \log \text{Tr} \rho_A^n \]
A common way to calculate Rényi entropies and other entanglement measurements is to exploit the replica trick [Calabrese, Cardy; 2004]

\[ C_n = \frac{l^{D-2}}{|\partial A|} \frac{\partial S_n}{\partial l} = \frac{l^{D-2}}{|\partial A|} \frac{1}{1 - n \alpha} \log \frac{Z_n(l + \alpha)}{Z_n(l)} \]
Jarzynski’s theorem [Jarzynski; 1996] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.

\[
\frac{Z_n(l + a)}{Z_n(l)} = \left\langle \exp \left( - \int \beta \delta W \right) \right\rangle
\]
Future work: $D = 3$ gauge $\mathbb{Z}_2$ and gauge-Higgs $\mathbb{Z}_2$

**Entropic c-function of the gauge Ising model**

- $n_x = 24, l_c = 6$
- $n_x = 32, l_c = 6$
- $n_x = 32, l_c = 8$
- $n_x = 48, l_c = 8$
- $n_x = 48, l_c = 10$
CP-broken Deconfined Phase at $\theta=\pi$

Shi Chen

*Department of Physics, the University of Tokyo*
4D SU(N) Yang-Mills (see Nati's lecture)  (pure, with adjoint matter, etc.)

- $\mathbb{Z}_N^{[1]}$ symmetry
  [Gaiotto, Kapustin, Seiberg, Willett, 2014]

\[
\begin{align*}
\begin{cases}
\text{Unbroken} & \rightarrow & \text{Confinement} \\
\text{Spontaneously broken} & \rightarrow & \text{Deconfinement/Higgsing}
\end{cases}
\end{align*}
\]

Couple background $\mathbb{Z}_N$ 2-form gauge field $w \in H^2(-, \mathbb{Z}_N)$

- Instanton charge fractionalization
  \[ \int c_2(a) + \frac{1}{N} \int P(w) = 0 \mod 1 \]

\[
\begin{align*}
\begin{cases}
N \text{ odd} & P(w) \equiv w \cup w \\
N \text{ even} & P(w) \equiv \frac{1}{2} P(w)
\end{cases}
\end{align*}
\]

Mixed 't Hooft anomaly

- Between $\mathbb{Z}_N^{[1]}$ and $\theta (U(1)^{1-1}$ symmetry)  $\rightarrow$  $\mathbb{Z}_N^{[1]}$ symmetric $\theta$ uniform phase
  [Córdova, Kapustin, Lam, Seiberg, 2019]

- At $\theta = \pi$: between $\mathbb{Z}_N^{[1]}$ symmetry and $CP$ ($\mathbb{Z}_2^{[0]}$ symmetry)  $\rightarrow$  $\mathbb{Z}_N^{[1]}$ symmetric $CP$ symmetric phase
  [Gaiotto, Kapustin, Komargodski, Seiberg, 2017]
$\xi$ is a parameter that causes a confinement-deconfinement phase transition

[Gaiotto, Kapustin, Komargodski, Seiberg, 2017]
$\mathcal{N} = 1$ SU(N) super Yang-Mills

- on $\mathbb{R}^2 \times S^1$ with susy boundary condition
- softly susy-breaking by a light massive gluino

$L$: the size of $S^1$

$m$: the mass of gluino

Semiclassical computation (see Mithat’s lecture)

[Davies, Hollowood, Khoze, 2000] [Poppotz, Schäfer, Ünsal, 2012]
[Chen, Fukushima, Nishimura, Tanizaki, 2020]

\[
\begin{align*}
(\phi, \sigma) & \in \frac{\mathbb{R}^r}{2\pi \Lambda_r^y} \times \frac{\mathbb{R}^r}{2\pi \Lambda_w^y} \\
(\phi, \sigma) & \xrightarrow{Z_N^{[1]}} (\phi + 2\pi \mu_c^y, \sigma) \\
\mu_c^y & \in \Lambda_w^y / \Lambda_r^y \simeq \mathbb{Z}_N
\end{align*}
\]

\[
\begin{cases}
\theta = 0: (\phi, \sigma) \xrightarrow{\text{CP}} (\phi, -\sigma) \\
\theta = \pi: (\phi, \sigma) \xrightarrow{\text{CP}} (\phi, -\sigma - \phi)
\end{cases}
\]

\[
V(\phi, \sigma) \propto \sum_{i=0}^r \sum_{j=0}^r M_i^+ (\alpha_i \cdot \alpha_j) M_j - \# \sum_{i=0}^r \frac{M_i^+ + M_i}{|\alpha_i|^2} \left( 1 - \frac{c_2 g^2}{8\pi^2} \ln |M_i| \right)
\]

\[
z \equiv i \left( \sigma + \frac{\theta}{2\pi} \phi \right) - \frac{4\pi}{g^2} \phi
\]

\[
\begin{cases}
M_i \equiv \exp \left\{ \alpha_i^y \cdot z + \frac{8\pi^2}{c_2 g^2} \right\} \text{ for } i = 1, 2, \ldots, r \\
M_0 \equiv \exp \left\{ \alpha_0^y \cdot z + \frac{8\pi^2 (1 - c_2)}{c_2 g^2} \right\} + i \theta
\end{cases}
\]
CP-breaking deconfinement phase

SU(N) with N > 2

SU(2)

[Chen, Fukushima, Nishimura, Tanizaki, 2020]
Thank you for your attention!
Leading Singularities of Gluon Amplitudes from the Bosonic String.

Carolina Figueiredo
July 2023

w/ Nina Arkani-Hamed
Qu Cao, Jin Dong
and Song He.

PITP
What are leading singularities?
What are leading singularities?

\[
\begin{align*}
\mathbf{p}_1 + \mathbf{p}_2 &= 0 \\
\mathbf{p}_4 + \mathbf{p}_5 &= 0 \\
\mathbf{p}^2_{\text{Internal}} &= 0
\end{align*}
\]
How to compute them?

Factorization:

1. $\epsilon_1^\mu$

2. $\epsilon_2^\mu$

3. $\epsilon_3^\mu$

4. $\epsilon_4^\mu$

5. $\epsilon_5^\mu$

$(p_1 \cdot p_2)^2 = 0$

$(p_3 \cdot p_4)^2 = 0$
How to compute them?

Factorization:

\[ 3\text{-point vertex} \]

\[ \alpha \delta_{\lambda \mu} \left[ g^{\lambda \nu} (k-p)^{\rho} + g^{\nu \rho} (p-q)^{\lambda} + g^{\rho \lambda} (q-k)^{\nu} \right] \]
\[ 3\text{-point vertex} \]

\[ \varphi \left( x^\mu \right) \varphi \left( x^{\nu} \right) \varphi \left( x^{\rho} \right) 
\times \delta^{x}_{y} \left[ g^{\mu \nu} \left( k - p \right)^{\rho} + g^{\nu \rho} \left( p - q \right)^{\mu} + g^{\rho \mu} \left( q - k \right)^{\nu} \right] \]

Increasing number of external gluons:
3-point vertex

\[ \propto \delta_{MN} \left[ g^\mu v (k-p)^\rho + g^v p (p-q)^\mu + g^p m (q-k)^v \right] \]

Increasing number of external gluons:

\[ N_{\text{vertices}} \rightarrow 3 \]
What else can we do?

Bosonic String.

$A^{\text{tra}}_6 = \int_{\mathcal{M}} \Omega$

[coarse-grained amplitudes are dark]
What else can we do?

Bosonic String.

[Diagram with labeled points 1 to 6 and intricate structure with arrows indicating $A_6$ and $\alpha' \to 0$.]
What else can we do?

Bosonic String.

\[ A_{\text{tree}} = \int \Omega \]

Res \[ \Omega \]

Leading Singularity

\[ \text{Res } \Omega \]
What else can we do?

Bosonic String:

\[ A_n^{(1,2,\ldots,n)} = \int \frac{dz_1 \ldots dz_n}{SL(2,\mathbb{R})} \prod_{i \neq j} \frac{2\pi^4}{z_{ij}} \exp \left\{ -\sum_{i \neq j} \left( \frac{\bar{x}^i E_i \cdot \bar{x}^j - 2 \bar{E}_i \cdot E_j}{z_{ij}} \right) \right\} \]

with \( z_{ij} = z_i - z_j \)

multi-linear in \( E_i \)
What else can we do?

Bosonic String.

\[ A_n^{+}(1,2,\ldots,n) = \int \frac{dZ_1 \ldots dZ_n}{SL(2,\mathbb{R})} \prod_{ij} z_{ij}^{2 \sum_{i} p_i^2 p_i^2} \exp \left\{ - \sum_{i \neq j} \left( \frac{\sqrt{x_i^2 x_j^2}}{z_{ij}} - 2 \frac{E_i E_j}{z_{ij}^2} \right) \right\} \]

\[ w/ \quad Z_{ij} = Z_i - Z_j \]

sum of lots of terms!

Ex.: \( A_6^{+} \), \( \sim 10^3 \) terms!
Bosonic String.

m-point Gluon amplitude
Trick: Gluons coming from scalars

\[ 2m \text{ Scalar Amplitude} \]
Scattering of $2n$ Scalars

\[ A^{\text{tree}}_{(1,2,\ldots,2n)} \propto \int \frac{dz_1 \cdots dz_{2n}}{\text{SL}(2,\mathbb{R})} \prod_{i<j} z_{ij} z_{i}^{*} z_{j}^{-} p_{i}^{\mu} p_{j}^{\mu} \times \frac{1}{z_{12}^2 z_{34}^2 \cdots z_{2n-1,2n}^2} \]

SINGLE TERM
Scattering of $2n$-Scalars

\[ A_{\text{tree}}^{(1, 2, \ldots, 2n)} \propto \int \frac{d^4 z_1 \cdots d^4 z_{2n}}{\text{SL}(2, \mathbb{R})} \prod_{i<j} \frac{1}{Z_{ij}^2} \left[ \frac{p_i \cdot p_j}{Z_{12}^2 Z_{34}^2 \cdots Z_{2n-1, 2n}^2} \right] \]

\[ A_{\text{true}}^{(1, \ldots, n)} [p_i \cdot p_f] \]

Single term

Residues on external gluons
\( m \) point gluon amplitude

A tree

\( \text{Residue on internal gluons} \)
$m$ point gluon amplitude

A tree

A gluons ($1, \ldots, m$)

Residue on internal gluons

$n$ point gluon

Leading Singularity
Loop level

2-m. scalar form for loop diagrams

\[ \text{Residues} \quad \left[ \neq \text{Bosonic string loop amplitudes} \right] \]

\[ \text{Gluon loop leading singularities} \]
Loop level

2-m. scalar form for loop diagrams \[ \text{SINGLE TERM!} \]

\[ \text{Residues \[ \neq \text{Bosonic string loop amplitudes} \]} \]

\[ \rightarrow \text{Gluon loop leading singularities!} \]

Thank You!
Confinement and Chiral Symmetry Breaking in AMSB QCD

Andrew Gomes
Cornell University

ArXiv: 2106.10288, 2107.02813, 2212.03260

With Csaba Csáki, Hitoshi Murayama, Ofri Telem,
Bea Noether, and Digvijay Roy-Varier
The goal

- NON-supersymmetric SU & SO gauge theories with fundamentals
- IR phase? Chiral symmetry breaking ($\chi$SB), quark confinement?
  - This “0th order” description is our aim

- Will achieve by augmenting the symmetry then removing it
A supersymmetric approach

- Augment QCD with gluinos (fermions) and squarks (bosons)
- IR hadrons are polynomials in the quark superfields
  - \( M = Q_L Q_R \) and \( B_{L/R} = Q_{L/R} \ldots Q_{L/R} \)
- And enough information about potential to fix many exotic IR phases!

- SUSY just a theoretical tool here!
The road to recovery

• SQCD interesting in its own right, but want real-world QCD → just give masses to superpartners, “theory deformation”
• Deformation must be identical in UV and IR descriptions
• Need something universal... Gravity!
  • Think about Einstein’s equivalence principle ($m_{\text{grav}} = m_{\text{inertial}}$)
Anomaly-mediated supersymmetry breaking

- In UV, only have loop effects:

\[ m_Q^2 = m_{\tilde{Q}}^2 = \frac{g^4}{(8\pi^2)^2} 2C_i(3N_c - N_f)m^2, \]

Effects in SU:

\[ m_\lambda = \frac{g^2}{16\pi^2}(3N_c - N_f)m. \]

- Exactly the theory we want when \( m \to \infty \) (for asymptotically free theories where \( 3N_c > N_f \))!
- But will always work with \( m < \Lambda \) to maintain perturbative control
- Now into the IR…

POMAROL, RATTazzi 9903448
Katz, Shadm, Shirman 9906296
Arkani-Hamed, Rattazzi 9804068
Giudice, Luty, Murayama, Rattazzi 9810442
Randall, Sundrum 9810155
For $N_c > N_f$, have Affleck-Dine-Seiberg (ADS) superpotential

\[
W = (N_c - N_f) \left( \frac{\Lambda^{3N_c-N_f}}{\det M} \right)^{1/(N_c-N_f)}
\]

Very different from non-SUSY QCD!

\[
Q = \bar{Q} = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
0 & \cdots & 0
\end{pmatrix}
\]

\[
M = \phi^2
\]

\[
V = \left| 2N_f \frac{1}{\phi} \left( \frac{\Lambda^{3N_c-N_f}}{\phi^{2N_f}} \right)^{1/(N_c-N_f)} \right|^2 - (3N_c - N_f)m \left( \frac{\Lambda^{3N_c-N_f}}{\phi^{2N_f}} \right)^{1/(N_c-N_f)} + c.c.
\]

Supersymmetric ADS potential

Flavor bifundamental gets a VEV \rightarrow Chiral symmetry breaking!
Phase collapse

- $N_f > 3N_c$
  - Theory IR free

- $N_f = 3N_c$
  - Conformal non-Abelian Coulomb phase
  - Free magnetic phase

- $N_f = 3/2N_c$
  - s-confinement
  - Quantum modified constraint
  - ADS superpotential, runaway vacuum

- $N_f = N_c + 1$
  - $N_f = N_c$
    - Global

- $N_f = 0$
  - $N_f > 3N_c$
  - AMSB
  - New!

- $N_f = 3/2N_c$
  - Chiral symmetry breaking

Csaki, AG, Murayama, Noether, Roy-Varier, Telem 2212.03260
The confinement story (SO)

- Why didn’t we talk about confinement?
  - Linear potential only until pair production
  - Cannot distinguish from Higgs phase

- Consider SO($N_c$) gauge theory with $N_f$ quarks in the vector representation
  - Also has spinor rep (e.g. spin-1/2 of $N_c = 3$)
  - Flux tube in this rep cannot be broken
  - We have a chance here!
The setup

- Consider the specific case \( N_f = N_c - 2 \) (N=1 family of SW analogs)
- SUSY theory has dual description (Seiberg dual) in terms of a U(1) gauge theory

\[
W_{\text{mon}} = \Lambda \left( \frac{\tilde{U}}{\Lambda^{N_F}} - 16 \right) \tilde{E}^+ \tilde{E}^- \\
\tilde{U} = \det \tilde{M}
\]

- Familiar meson \( M \), and U(1) electric charges \( E^\pm \)
  - Though not a true electric/magnetic duality, can think of as magnetic monopoles of original theory
A twin condensation

- As before, add AMSB:

\[
\tilde{M} = 16 \frac{1}{N_F} \Lambda, \quad |\tilde{E}^+||\tilde{E}^-| = 16 \frac{2}{N_F}^{-1} k m \Lambda
\]

Usual meson condensate → Chiral symmetry breaking!
Monopole condensate → Confinement!

- First demonstration of confinement and continuous chiral symmetry breaking in a non-supersymmetric gauge theory

- Can extend results to \( 0 < N_f < 3(N_c - 2) \)
  - Same phase collapse seen in SU
Towards a string theory for 2d Yang Mills

Tal Sheaffer

Weizmann Institute of Science

13/07/23

Ongoing work with Ofer Aharony and Suman Kundu (WIS)
Gross-Taylor Series

- 2d YM exactly solvable (Kazakov & Kostov (1980), Rusakov (1990), Witten (1991) & more)
- Exact large $N$ expansion of observables $Z, \langle WL \rangle$ (Gross-Taylor 1993)
- Organizes into sums of worldsheet maps
- No worldsheet CFT / action
- Only particular maps contribute (supersymmetric localization?)
Worldsheet proposals with supersymmetric localization

- **Topological** string theory for 0-coupling YM.
- Localization to **holomorphic** maps \( \text{Cordes, Moore, Ramgoolam (1994)} \)
  - Only “chiral” contributions
- Localization to **extremal area** maps (soap films!) \( \text{Hořava (1996)} \)
  - Solutions of **Nambu-Goto** EOM
  - Includes “non-chiral” maps
- Vague proposals for finite ’t Hooft coupling \( \lambda \)
The devil’s in the details

- A term in Hořava’s action \textit{vanishes identically} \( \Rightarrow \) ill defined moduli-space integral.
  - We found a gauge-invariant non-vanishing replacement \( \Rightarrow \) correct measure on moduli space for \( \lambda = 0 \) (topological Yang Mills).

- Gauge fixing and path integration in Hořava’s Nambu-Goto-type theory ill-behaved.
  - We reformulated as a Polyakov-type path-integral
  - Naturally regulates some (hopefully all) the ill-behaved non-chiral maps.

- What about Wilson-loops?
  - We adapted the theory to worldsheets with boundaries.
Down the road

- Obtain the correct measure on the boundary of moduli space, for maps without boundaries.
- Compute more amplitudes.
- Generalize to nonzero 't Hooft coupling $\lambda$ - too early for optimism.
- Stringy derivation of 't Hooft spectrum of mesons
- More speculative:
  - Adjoint particles as weights on the string
  - Higher $D$
Thanks!
Applications of the modified Villain formulation

Theo Jacobson, University of Minnesota

Based on 2303.06160 with Tin Sulejmanpasic and work in progress
The modified Villain formulation

Remove lattice-scale topological defects (vortices, monopoles) to endow lattice field theories with features of their continuum limits:

- Global symmetries
- 't Hooft anomalies
- Dualities

Villain: $a_\ell \in \mathbb{R}, \ n_p \in \mathbb{Z}, \ a_\ell \rightarrow a_\ell + 2\pi m_\ell, \ n_p \rightarrow n_p + (dm)_p$

Monopole: $(dn)_c \neq 0$

Modified Villain: constrain $(dn)_c = 0$

(Gross, Klebanov '90, Gattringer, Sulejmanpasic '19, Gorantla, Lam, Seiberg, Shao '21)
Abelian Chern-Simons theory

Goal: discretize $U(1)_k$ CS theory on a Euclidean spacetime lattice, with:

- large gauge invariance + level quantization
- $\mathbb{Z}_k$ 1-form symmetry + 't Hooft anomaly
- electric charge of monopoles
- framing

Application: establish boson/fermion dualities nonperturbatively

- extend exact lattice particle-vortex duality
Framing and spin of Wilson lines

Minimal charge Wilson loops are **ribbons**: edges connected by a surface

(ordinary Wilson loops are projected out by a peculiar gauge redundancy)

Compute topological spin $s_q = \frac{q^2}{2k}$ via self-linking
Axion-Maxwell theory

Continuum action: \[ \mathcal{L} = \frac{iK}{8\pi^2} \theta F \wedge F, \quad K \in \mathbb{Z} \]

Symmetries:

- \( \mathbb{Z}_K^{(0)} \) axion shift symmetry
- \( \mathbb{Z}_K^{(1)} \) acts on Wilson loops
- \( U(1)_m^{(1)} \) acts on 't Hooft loops
- \( U(1)_s^{(2)} \) acts on axion strings
- ... 

Higher group structure:
Tanizaki, Ünsal '19
Hidaka, Nitta, Yokokura '20
Brennan, Córdova '20
Choi, Lam, Shao '22

Goal: preserve and study these symmetries on the lattice
Higher-group symmetry and axion strings

Generator of $U(1)_s^{(2)}$

Axion string
worldsheet
+ chiral boson

Generators of $\mathbb{Z}_K^{(1)}$

Wilson line

$e^{\frac{2\pi i}{K}}$
Holomorphic Confinement of $\mathcal{N} = 1$ SYM

PiTP 2023

Justin Kulp
Kasia Buddzik, Davide Gaiotto, Brian Williams, Jingxiang Wu, Matt Yu.

Perimeter Institute for Theoretical Physics

13/Jul/2023

arXiv:2207.14321
arXiv:2306.01039
Given any SQFT, we obtain the **Holomorphic Twist** by taking cohomology of any nilpotent supercharge, e.g. $Q := Q_-$. Anti-holomorphic translations are $Q$-exact, so twisted theory is (cohomologically) holomorphic [Johansen], [Witten], [Costello]

\[
\{Q, \bar{Q}_\alpha\} = \partial_{\bar{z}^\alpha}
\]  

(1)
Holomorphic Twists and Infinite Symmetries

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\[ \{Q, \bar{Q}_\alpha\} = \partial\bar{z}^\alpha \quad (1) \]

- Protected subsector of **semi-chiral operators**, i.e. $[Q, \mathcal{O}] = 0$
  - In SCFTs: includes those counted by superconformal index

\[ I = \text{Tr}(-1)^F p^{j_1+j_2-r/2} q^{j_1-j_2-r/2} e^{-\beta\{Q_-, S_+\}} \quad (2) \]
Holomorphic Twists and Infinite Symmetries

- Given any SQFT, we obtain the **Holomorphic Twist** by taking cohomology of any nilpotent supercharge, e.g. $Q := Q_-$. 
  - Anti-holomorphic translations are $Q$-exact, so twisted theory is (cohomologically) holomorphic [Johansen], [Witten], [Costello]

$$\{Q, \bar{Q}_\alpha\} = \partial_{\bar{z}^\alpha}$$  \hspace{1cm} (1)

- Protected subsector of **semi-chiral operators**, i.e. $[Q, \mathcal{O}] = 0$
  - In SCFTs: includes those counted by superconformal index

$$\mathcal{I} = \text{Tr}(-1)^F p^{j_1 + j_2 - r/2} q^{j_1 - j_2 - r/2} e^{-\beta\{Q_-, S^\pm\}}$$  \hspace{1cm} (2)

- Theories are equipped with local product called **$\lambda$-Bracket**

$$\{\mathcal{O}_1, \mathcal{O}_2\}_\lambda = \int_{S^3} e^{\lambda \cdot z} d^2 z \mathcal{O}_1(z) \mathcal{O}_2(0)$$  \hspace{1cm} (3)

  - Infinite dimensional symmetry enhancements analogous to Virasoro and Kac-Moody, but in 4d [Gwilliam, Williams].
  - Higher brackets $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{n+1}\}_{\lambda_1, \lambda_2, \ldots, \lambda_n}$
Feynman diagrams must be **Laman graphs**.

- Change of variables maps Feynman integrals to a polytope in space of holomorphic loop momenta: the **operatope**.
- Recursive identities of the operatope used to **bootstrap** Feynman integrals to at least 3-loops (perhaps further!)
Cohomologies and Feynman Diagrams

- Feynman diagrams must be Laman graphs.

- Change of variables maps Feynman integrals to a polytope in space of holomorphic loop momenta: the operatope.

- Recursive identities of the operatope used to bootstrap Feynman integrals to at least 3-loops (perhaps further!)

- Polynomials in fields and derivatives \( \leadsto \) Free Cohomology \( \mathcal{V} \)
  - Interacting quantum theory is obtained from underlying free-classical theory \( \mathcal{V} \) as cohomology of a new operator

\[
Q = Q_0 + Q_1 + Q_2 + \ldots
\]  

where \( Q_n \) is computed by \( n \)-loop Feynman diagrams.

- All perturbative corrections are contained in the brackets!

\[
Q \mathcal{O} = \{I, \mathcal{O}\}_0 + \{I, I, \mathcal{O}\}_{0,0} + \{I, I, I, \mathcal{O}\}_{0,0,0} + \ldots
\]
\( \mathcal{N} = 1 \text{ SYM} \) is \( SU(N) \) gauge theory with an adjoint fermion

\[
\mathcal{L} = \int d^2 \theta \frac{-i}{8\pi} \tau \text{ tr } W_\alpha W^\alpha + \text{c.c.}
\] (6)

- Twist is identified (in a non-trivial way) with a \( bc \) system
- Free cohomology is gauge invariant polynomials in \( b, c, \) and \( \partial \).
\( \mathcal{N} = 1 \, \text{SYM} \) is \( SU(N) \) gauge theory with an adjoint fermion

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Adding **one-loop corrections** changes the cohomology: we find the surviving stress tensor becomes \( Q \)-exact.
- Theory becomes **topological** at one loop!
\( \mathcal{N} = 1 \) SYM is \( SU(N) \) gauge theory with an adjoint fermion

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- Twist is identified (in a non-trivial way) with a \textit{bc system}.
- Free cohomology is gauge invariant polynomials in \( b, c, \) and \( \partial \).

Adding \textbf{one-loop corrections} changes the cohomology: we find the surviving stress tensor becomes \( Q \)-exact.
- Theory becomes \textbf{topological} at one loop!

Being topological is compatible with \textbf{confinement}: if topological in the UV, then topological in the IR.
- Constrains IR physics: the holomorphic twist of the IR must also be topological.
Trace relations and open string vacua

Ji Hoon Lee

2109.02545 D. Gaiotto, JHL
2204.09286 JHL
Work to appear
A recurring theme in our study of holography is the idea that the Feynman diagrams of a large $N$ gauge theory reorganize into a genus expansion of some string theory.

It is natural to ask to what extent the notion of gauge-string duality persists at finite $N$.

I will explain an intriguing pattern, called the giant graviton expansion, that appears in the spectrum of states of finite $N$ gauge theories on $S^1 \times S^3$ at zero coupling $\lambda = 0$. 
Namely, the finite $N$ spectrum of a $U(N)$ gauge theory reorganizes into a systematic set of $e^{-kN}$ corrections to the large $N$ spectrum:

$$Z_{U(N)} = Z_{U(\infty)} \sum_{k=0}^{\infty} (-1)^k x^{kN} \tilde{Z}_{U(k)}.$$ 

These corrections have transparent holographic interpretations in the bulk — they are contributions from $k$ giant graviton branes and open/closed string excitations thereof.
The giant graviton expansion is a formula that seems to suggest rules in the bulk that we did not know before.

\[ Z_{U(N)} = Z_{U(\infty)} \sum_{k=0}^{\infty} (-1)^k x^{kn} \hat Z_{U(k)} . \]

For example, why is the full bulk partition function in the \( \alpha' \to \infty \) limit given by a sum over brane sectors?

Also, why does the sum over brane sectors exhibit huge cancellations to give \( Z_{U(N)} \), even when we are computing the free partition function rather than the index?
Recall that giant gravitons were originally discovered to provide a bulk explanation for the presence of finite $N$ trace relations. Thus, let us answer these bulk questions by revisiting how one implements trace relations in a $U(N)$ gauge theory.

The Hilbert space of a free finite $N$ gauge theory is given by the highly-constrained quotient module

$$M_N = M_\infty / \langle \text{trace relations at } N \rangle$$

of the infinite $N$ Hilbert space $M_\infty$ modulo the ideal of trace relations, over the ring

$$R = \mathbb{C}[\text{Tr } X, \text{Tr } XY, \text{Tr } \psi F... , \text{Tr } \partial \partial \partial X, \cdots]$$

of all gauge invariant polynomials of fields and their derivatives at infinite $N$. 
A free resolution of $M_N$

$$\cdots \rightarrow V_3 \overset{\hat{Q}}{\rightarrow} V_2 \overset{\hat{Q}}{\rightarrow} V_1 \overset{\hat{Q}}{\rightarrow} M_\infty \rightarrow M_N \rightarrow 0$$

replaces $M_N$ with an infinite exact sequence of free modules $V_k$ with differential $\hat{Q}$.

$V_k$ is the space of $k$-th order relations among the generators of $M_\infty$, with the $k = 1$ case $V_1$ being the space of trace relations.

We can view the free resolution of $M_N$ as the procedure of introducing "heavy" ghosts for a $U(\infty)$ theory that compensate for null states due to trace relations at some $N$. 
I would now like to argue the following (modulo instantons):

\[ V_k \simeq \mathcal{H}_\infty^{\text{closed}} \otimes \mathcal{H}_k^{\text{open}} \]

The free module \( V_k \) of \( k \)-th order trace relations in a \( U(N) \) gauge theory should be interpreted in the string dual as the \( \alpha' \to \infty \) limit of the space of open string states on \( k \) giant graviton branes sharing a closed-string background.

(This is based on matches of the hilbert series of \( V_k \) with the \( k \)-th term in the giant graviton expansion, for \( \frac{1}{2} \) and \( \frac{1}{4} \)-BPS sectors of \( \mathcal{N} = 4 \) SYM.)

The giant graviton expansion

\[ Z_{U(N)} = Z_{U(\infty)} \sum_{k=0}^{\infty} (-1)^k x^{kN} \hat{Z}_{U(k)}. \]

is then the refined Euler characteristic associated to a free resolution graded by the “heavy ghost” number.
Thank you