ARITHMETIC RAMSEY THEORY
PROBLEM SET # 1

Problems marked with a $\star$ are harder.

(1) Recall the finitary statement of van der Waerden's theorem from the lecture:

Theorem 1. *Let $k, r \in \mathbf{N}$. There exists $W = W(k,r) \in \mathbf{N}$ such that, whenever $N \geq W$, any r-coloring of $[N]$ contains a monochromatic k-term nontrivial arithmetic progression.*

and the infinitary statement:

Theorem 2. *If $\mathbf{N}$ is finitely colored, then some color class contains nontrivial arithmetic progressions of all finite lengths.*

(a) Prove that Theorem 1 implies Theorem 2. (Hint: First show that for any fixed $k \in \mathbf{N}$, if $\mathbf{N}$ is finitely colored, then some color class contains a nontrivial k-term arithmetic progression. Then, use the pigeonhole principle.)

(b) Prove that Theorem 2 implies Theorem 1. (Hint: Consider the space $[r]^{\mathbf{N}}$ of all r-colorings of $\mathbf{N}$ equipped with the product topology (where each $[r]$ is given the discrete topology). This is compact and metrizable, and hence sequentially compact.)

(2) Construct a 2-coloring of $\mathbf{N}$ in which there is no monochromatic infinite arithmetic progression.

(3) Prove that Szemerédi's theorem implies van der Waerden's theorem.

(4) Prove that for all $A, B, C \subset \mathbf{F}_p$ with $|A||B||C| \gg p^{11/4}$ there exists $x, y \in \mathbf{F}_p$ such that $(x, x+y, xy) \in A \times B \times C$ assuming the following result:

Theorem 3. *Let $f, g, h : \mathbf{F}_p \to \mathbf{C}$ be 1-bounded. Then, $\Lambda(f,g,h)$ equals*

$$\left(\mathbf{E}_{x\in\mathbf{F}_p} f(x)\right)\left(\mathbf{E}_{x\in\mathbf{F}_p} g(x)\right)\left(\mathbf{E}_{x\in\mathbf{F}_p} h(x)\right) + O\left(p^{-1/4}\right).$$

(5) Set

$$\mu(\mathbf{c}) = \frac{\#\{(x,a,y,z) \in \mathbf{F}_p^n : (x+ya, xa+y, x+za, xa+z) = \mathbf{c}\}}{p^4}$$

for all $\mathbf{c} \in \mathbf{F}_p^4$ and let $\tilde{\mu}$ denote the uniform measure on $\mathbf{F}_p^4$. Let $\mathbf{c} \in \mathbf{F}_p^4$. Prove that

(a) whenever $c_2 \neq c_4$ and $(c_1 - c_3) \neq \pm(c_2 - c_4)$, $\mu(\mathbf{c}) = p^{-4}$,

(b) whenever $c_2 = c_4$, $\mu(\mathbf{c}) = 0$ if $c_1 \neq c_3$ and $\mu(\mathbf{c}) = p^{-3} - 2p^{-4}$ if $c_1 = c_3$, and

(c) whenever $c_2 \neq c_4$ and $(c_1 - c_3) = \pm(c_2 - c_4)$, $\mu(\mathbf{c}) = p^{-3}$ if $(c_1, c_3) = (c_2, c_4)$ or $(c_1, c_3) = (-c_2, -c_4)$ and $\mu(\mathbf{c}) = 0$ otherwise.

Then, deduce that

$$\sum_{\mathbf{c} \in \mathbf{F}_p^4} |\mu(\mathbf{c}) - \tilde{\mu}(\mathbf{c})| \ll \frac{1}{p}.$$

(6) Re-run the argument from the lecture more carefully to prove that for all $A, B, C \subset \mathbf{F}_p$ with $|A||B||C| \gg p^{5/2}$ there exists $x, y \in \mathbf{F}_p$ such that $(x, x+y, xy) \in A \times B \times C$.

(7) ($\star$) Prove the non-linear Roth theorem of Bourgain and Chang (possibly with a worse, but still power-saving, exponent) using the ideas from the lecture.