EXERCISES PCMI

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1. The Grothendieck-Witt ring of a field k

Recall from the lecture that the Grothendieck-Witt ring of a field k is the group completion of isometry classes of non-degenerate symmetric bilinear forms on finite dimensional k-vector spaces (or equivalently if char $k \neq 2$ non-degenerate quadratic forms) over k. Since any such form can be diagonalized, $\mathrm{GW}(k)$ has generators

$$\langle a \rangle : k \times k \to k,\ (x, y) \mapsto axy$$

where $a \in k^\times$ with relations

(1) $\langle a \rangle = \langle ab^2 \rangle$ for $a, b \in k^\times$
(2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $a, b \in k^\times$
(3) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $a, b \in k^\times$ with $a + b \neq 0$
(4) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$ for $a \in k^\times$.

Exercise 1.1. Show that the fourth relation follows from first three.

Exercise 1.2. Show that the first three relations make sense, that is, show that the forms represent the same class in $\mathrm{GW}(k)$.

2. $\mathbb{A}^1$-degree

Let k be a field.

2.1. Endomorphisms of $\mathbb{P}^1$.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ and let $p \in \mathbb{P}^1(k)$ be a rational point such that $f^{-1}(p) = \{q_1, \dots, q_s\}$ and $J(q_i) := f'(q_i) \neq 0$ for $i = 1, \dots, s$. We have seen in the lecture that the $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}(f)$ of f can be calculated as the following sum

$$\deg^{\mathbb{A}^1}(f) = \sum_{i=1}^{s} \langle J(q_i) \rangle \in \mathrm{GW}(k).$$

Exercise 2.1. Compute the $\mathbb{A}^1$-degree of the following maps

(1) $x \mapsto ax$
(2) $x \mapsto x^2$
(3) $x \mapsto \frac{x^2-1}{x}$.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism of $\mathbb{P}^1$ of the form $f = \frac{A(x)}{B(x)}$ with polynomials $A = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and $B = b_{n-1}x^{n-1} + \ldots + b_0$ with no common zeros.

Then

$$\frac{A(X)B(Y) - A(Y)B(X)}{X - Y} \in k[X, Y]$$

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can be written as

$$\frac{A(X)B(Y)-A(Y)B(X)}{X-Y}=\sum_{1\leq i,j\leq n} c_{i,j}X^{i-1}Y^{j-1}$$

for $c_{i,j}\in k$. The matrix $(c_{i,j})$ is symmetric and non-degenerate and thus determines an element of $\mathrm{GW}(k)$ which is equal to the $\mathbb{A}^1$-degree of f by a result of Cazanave [Caz12] .

Exercise 2.2. Use Cazanave's formula to determine the $\mathbb{A}^1$-degree of

(1) $x\mapsto ax$
(2) $x\mapsto x^2$
(3) $x\mapsto \frac{x^2-1}{x}$.

2.2. **Endomorphisms of $\mathbb{A}^n$.** Let $f=(f_1,\dots,f_n):\mathbb{A}^n\to\mathbb{A}^n$ be an endomorphism of $\mathbb{A}^n$ with only isolated zeros. In particular, $Q:=\frac{k[x_1,\dots,x_n]}{(f_1,\dots,f_n)}$ is a finite dimensional k-vector space. We introduce new variables $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_n)$ and define

$$\Delta_{i,j}:=\frac{f_i(Y_1,\dots,Y_{j-1},X_j,\dots,X_n)-f_i(Y_1,\dots,Y_j,X_{j+1},\dots,X_n)}{X_j-Y_j}$$

and let $\mathrm{Béz}(f_1,\dots,f_n)$ be the image of $\det\Delta_{i,j}$ in $\frac{k[X,Y]}{(f(X),f(Y))}$. Choose a k-basis $a_1,\dots,a_m$ of $Q=\frac{k[x_1,\dots,x_n]}{(f_1,\dots,f_n)}$ and express $\mathrm{Béz}(f_1,\dots,f_n)$ as

$$\mathrm{Béz}(f_1,\dots,f_n)=\sum_{1\leq i,j\leq m} c_{i,j}a_i(X)a_j(Y).$$

Then the matrix $(c_{i,j})$ is symmetric and non-degenerate and calculates the $\mathbb{A}^1$-degree of f by [BMP21].

Exercise 2.3. Show that the formula of the $\mathbb{A}^1$-degree of an endomorphism of $\mathbb{A}^n$ agrees with Cazanave's formula when $n=1$.

Exercise 2.4. Let $f:\mathbb{A}^{n_1}\to\mathbb{A}^{n_1}$ and $g:\mathbb{A}^{n_2}\to\mathbb{A}^{n_2}$. Show that $\deg^{\mathbb{A}^1}(f\times g:\mathbb{A}^{n_1+n_2}\to\mathbb{A}^{n_1+n_2})$ is equal to the product $\deg^{\mathbb{A}^1}(f)\cdot\deg^{\mathbb{A}^1}(g)$.

Exercise 2.5. Calculate the $\mathbb{A}^1$-degree of

(1) $x\mapsto ax^d$ for an integer $d>0$
(2) $(x_1,\dots,x_n)\mapsto(a_1x_1^{d_1},\dots,a_nx_n^{d_n})$ for positive integers $d_1,\dots,d_n$

3. Enumerative counts in GW(k)

We have seen in the lecture that one can count geometric objects as solutions to $f=0$ where f is a section of a vector bundle $p:V\to X$ where X is smooth and proper over k

$$e(V)=e(V,f)=\sum_{q\in f=0}\deg_q^{\mathbb{A}^1} f\in\mathrm{GW}(k).$$

Whenever $\mathrm{rank}\, V=\dim X$, $\{f=0\}$ is zero dimensional and the bundle V is "sufficiently oriented" the count $e(V)$ is independent of the choice of f.

3.1. **Lines meeting four lines in $\mathbb{P}^3$.** One can for example count lines in $\mathbb{P}^3$ meeting four given lines this way (see [SW21]). Let $\mathrm{Gr}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3$ (or equivalently 2-planes in 4-space) and let $\mathcal{S} \to \mathrm{Gr}(1,3)$ be the tautological bundle.

Exercise 3.1. Let L be the line in $\mathbb{P}^3$ cut out by two linear forms $\alpha, \beta \in k[X_0, X_1, X_2, X_3]$. Then $\alpha \wedge \beta$ defines a section of $\bigwedge^2 \mathcal{S}^\vee \to \mathrm{Gr}(1,3)$. Show that the zeros of the section $\alpha \wedge \beta$ correspond to the lines in $\mathbb{P}^3$ meeting L.

It follows from the exercise that the zeros of a section of $V := \bigoplus_{i=1}^4 \bigwedge^2 \mathcal{S}^\vee \to \mathrm{Gr}(1,3)$ count lines meeting four given lines in $\mathbb{P}^3$.

Let U be the open subset of lines in $\mathrm{Gr}(1,3)$ of the form $[s:t] \mapsto [xs + x't : ys + y't : s : t]$. Then U can be identified with $\mathbb{A}^4 = \mathrm{Spec}(k[x, x', y, y'])$ and the bundle V trivializes over U and a section $f : \mathrm{Gr}(1,3) \to V$ restricts to $f|_U : \mathbb{A}^4 \to V|_U \cong U \times \mathbb{A}^4$ which defines an endomorphism of $\mathbb{A}^4$ which we also call $f : \mathbb{A}^4 \to \mathbb{A}^4$.

If the chosen section f of V has finitely many zeros all lying in U one can compute $e(V)$ as the $\mathbb{A}^1$-degree of $f : \mathbb{A}^4 \to \mathbb{A}^4$.

Exercise 3.2 (Example 6.1 in [SW21])**.** Assume $\mathrm{char}\, k \neq 2, 3, 5$ and let $L_1 = \{X_1 = X_2\} \subset \mathbb{P}^3$, $L_2 = \{X_0 - X_2 = X_1 - X_3 = 0\} \subset \mathbb{P}^3$, $L_3 = \{X_1 - 2X_3 = 2X_0 - X_2 = 0\} \subset \mathbb{P}^3$ and $L_4 = \{X_0 - 3X_1 = 3X_2 - X_3 = 0\} \subset \mathbb{P}^3$. What are the lines meeting L_1, L_2, L_3 and L_4? Find the corresponding section f of V and compute the $\mathbb{A}^1$-degree of $f : \mathbb{A}^4 \to \mathbb{A}^4$. Conclude that the count of lines in $\mathbb{P}^3$ meeting four given lines is $\langle 1 \rangle + \langle -1 \rangle \in \mathrm{GW}(k)$. What does this tell you about the count over $k = \mathbb{C}$ and $k = \mathbb{R}$?

3.2. **Bézout's theorem over arbitrary fields k [McK21].** Let $V := \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(d_n) \to \mathbb{P}^n$ for positive integers $d_1, \ldots, d_n$. Then one can count the points in the intersection of the zero loci of $f_1, \ldots, f_n$ where f_i is a homogeneous polynomial in $X_0, \ldots, X_n$ of degree d_i for $i = 1, \ldots, n$, as the zeros of the section of V defined by $f_1, \ldots, f_n$.

Let $T\mathbb{P}^n$ be the tangent bundle over $\mathbb{P}^n$. Then V is "sufficiently orientable" if there exists a line bundle $\mathcal{L}$ on $X = \mathbb{P}^n$ and an isomorphism $\mathrm{Hom}(\det T\mathbb{P}^n, \det V) \cong \mathcal{L}^{\otimes 2}$.

Exercise 3.3. For which n and $d_1, \ldots, d_n$ is V "sufficiently orientable"?

Exercise 3.4. Compute $e(V)$ for n and $d_1, \ldots, d_n$ for which V is "sufficiently orientable". Hint: You can use Exercise 2.5.

References

[BMP21] Thomas Brazelton, Stephen McKean, and Sabrina Pauli. Bézoutians and the $\mathbb{A}^1$-degree, 2021.

[Caz12] Christophe Cazanave. Algebraic homotopy classes of rational functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(4):511–534 (2013), 2012.

[McK21] Stephen McKean. An arithmetic enrichment of bézout's theorem. *Mathematische Annalen*, 379(1-2):633–660, Jan 2021.

[SW21] Padmavathi Srinivasan and Kirsten Wickelgren. An arithmetic count of the lines meeting four lines in $\mathbf{P}^3$. *Trans. Amer. Math. Soc.*, 374(5):3427–3451, 2021. With an appendix by Borys Kadets, Srinivasan, Ashvin A. Swaminathan, Libby Taylor and Dennis Tseng.

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