NOTE ON THE NUMBER OF BALANCED INDEPENDENT SETS IN THE HAMMING CUBE

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ABSTRACT. Let Q_d be the d-dimensional Hamming cube and $N = |V(Q_d)| = 2^d$. An independent set I in Q_d is called balanced if I contains the same number of even and odd vertices. We show that the logarithm of the number of balanced independent sets in Q_d is

$$(1 - \Theta(1/\sqrt{d}))N/2.$$

The key ingredient of the proof is an improved version of "Sapozhenko's graph container lemma."

1. Introduction

For a bipartite graph $G = X \coprod Y$ and an independent set I in G, I is said to be *balanced* if $|I \cap X| = |I \cap Y|$. We use bis(G) for the number of balanced independent sets (BIS's) of a graph G.

Write Q_d for the d-dimensional Hamming cube and N for $|V(Q_d)| (= 2^d)$. In this note we prove the following result on $\log \operatorname{bis}(Q_d)$. (All \log 's in this paper are in base 2.)

Theorem 1.1.

(1)
$$\log \operatorname{bis}(Q_d) = (1 - \Theta(1/\sqrt{d}))N/2.$$

It is easy to see that the rhs of (1) is a lower bound: Barber [2] showed that the size of a maximum BIS in Q_d is

$$\begin{cases} 2^{d-1} - 2 \binom{d-2}{(d-2)/2} & \text{if } d \text{ is even;} \\ \\ 2^{d-1} - \binom{d-1}{(d-1)/2} & \text{if } d \text{ is odd,} \end{cases}$$

and collecting balanced subsets of a maximum BIS gives the lower bound. So the main task of this paper is to show the rhs of (1) is also an upper bound.

Background. The asymptotics for the number of (ordinary) independent sets in Q_d , $i(Q_d)$, was first given by Korshunov and Sapozhenko [10]:

Theorem 1.2.

$$i(Q_d) \sim 2\sqrt{e}2^{N/2}.$$

(The above asymptotics are substantially refined by Jenssen and Perkins in [7].) Note that the rhs of (2) is an asymptotic lower bound on $i(Q_d)$: writing $Q_d = \mathcal{E} \coprod \mathcal{O}$ (a few basic definitions are recalled below), any

The author is supported directly by NSF grant DMS-1926686 and indirectly by NSF grant CCF-1900460.

subset of \mathcal{E} or \mathcal{O} is an independent set, from which we have $2 \cdot 2^{N/2} - 1$ independent sets. The extra factor \sqrt{e} reflects the contribution of independent sets most of whose vertices are even (odd, resp.), together with a (very) small number of odd (even, resp.) vertices. (See e.g. [4] for a more detailed description on this lower bound construction.)

Thus Theorem 1.2 implies that $i(Q_d)$ is asymptotically equal to this lower bound, and in particular, this implies all but a negligible fraction of independent sets in Q_d are highly unbalanced. The natural problem of estimating $bis(Q_d)$ was suggested by T. Helmuth, M. Jenssen, and W. Perkins [5], and Theorem 1.1 answers this question at the level of asymptotics of the logarithm.

The key ingredient of the proof of Theorem 1.1 is Lemma 3.1, an improvement (see Remark 3.2) of "Sapozhenko's graph container lemma" from [11]. Sapozhenko's lemma and its variants have played a key role in resolving a number of asymptotic enumeration problems on the Hamming cube and related structures, e.g. [10, 7, 3, 8, 9, 6, 1]. The current improved version of the lemma is implicitly proved in [9, Lemma 6.3], but we give a self-contained proof in Section 3 to provide a convenient reference for future work.

Definitions. We use Q_d for the d-dimensional Hamming cube: that is, $V = V(Q_d)$ is the collection of binary strings of length d, and two vertices are adjacent iff they differ in exactly one coordinate. A vertex v is even (odd, resp.) if v contains an even (odd, resp.) number of 1's. We use \mathcal{E} (\mathcal{O} , resp.) for the set of even (odd, resp.) vertices in Q_d (so $Q_d = \mathcal{E} \coprod \mathcal{O}$). The collection of balanced independent sets (BIS's) in Q_d is denoted by $\mathcal{B} = \mathcal{B}(d)$, and I always denotes a BIS.

As usual, N(v) is the set of neighbors of v, and N(A) is the set of vertices that are adjacent to at least one vertex in A. We use [A] for the *closure* of A, namely, $[A] = \{v \in V : N(v) \subseteq N(A)\}$.

Finally, we refer to the logarithm of the number of possibilities for a choice as the *cost* of that choice.

Outline. In Section 2 we recall some basic tools. The main lemma (Lemma 3.1) and Theorem 1.1 are proved in Section 3 and Section 4 respectively.

2. Tools

The following is a well-known fact about the sum of binomial coefficients.

Proposition 2.1. For any fixed $\alpha \in [0, 1/2]$ and $n \in \mathbb{Z}^+$,

$$\sum_{i \le \alpha n} \binom{n}{i} \le 2^{H(\alpha)n},$$

where $H(\alpha) := -\alpha \log \alpha - (1 - \alpha) \log (1 - \alpha)$ is the binary entropy function.

For a positive integer m, a *composition* of m is a sequence (a_1, \ldots, a_s) of positive integers summing to m. Recall the following basic fact:

Proposition 2.2. The number of compositions of m is 2^{m-1} and the number with at most $b \le m/2$ parts is

$$\sum_{i < b} \binom{m-1}{i} \le \exp_2[b \log(em/b)].$$

Say $A \subseteq V$ is 2-linked if for any $u, v \in A$, there are vertices $u = u_0, u_1, \dots, u_l = v$ in A such that for each $i \in [l]$, u_{i-1} and u_i are at distance at most 2 in Q_d . The 2-components of A are its maximal 2-linked subsets.

Proposition 2.3 ([3], Lemma 1.6). For each fixed k, the number of k-linked subsets of V of size x containing some specified vertex is at most $2^{O(x \log d)}$.

The next two results recall standardish isoperimetric inequalities for Q_d . Recall that $N = |V(Q_d)| = 2^d$.

Proposition 2.4 ([4], Claim 2.5). *For* $A \subseteq \mathcal{E}$ (or \mathcal{O}) with $|A| \leq N/4$,

$$\frac{|N(A)| - |A|}{|N(A)|} = \Omega(1/\sqrt{d}).$$

Proposition 2.5 ([4], Lemma 2.6). *For* $A \subseteq \mathcal{E}$ (or \mathcal{O}),

if
$$|A| < d^{O(1)}$$
, then $|A| \le O(1/d)|N(A)|$.

The next lemma recalls what we need from [11], and follows from Lemmas 5.3-5.5 in the excellent exposition due to Galvin [4]. For A in the statement, we use G = N(A) and t = |G| - |A|.

Lemma 2.6. For $q, g \in \mathbb{Z}^+$, $q \leq N/4$, $g \geq d^4$ and

$$\mathcal{G}(q,g) = \{ A \subseteq \mathcal{E} : A \text{ is 2-linked, } |[A]| = q \text{ and } |G| = g \},$$

there exist a family $W = W(q, g) \subseteq 2^{\mathcal{E}} \times 2^{\mathcal{O}}$ with

$$|\mathcal{W}| = 2^{O(t \log^2 d/\sqrt{d})}$$

and a function $\Phi = \Phi_{q,q} : \mathcal{G} \to \mathcal{W}$ such that for each $A \in \mathcal{G}$, $(S,F) := \Phi(A)$ satisfies:

- (a) $S \supseteq [A], F \subseteq G$;
- (b) $|S| \le |F| + O(t/(\sqrt{d}\log d))$.

3. MAIN LEMMA

In this section we prove the following key lemma.

Lemma 3.1. For q, g, and $\mathcal{G}(q, g)$ as in Lemma 2.6,

$$\log |\mathcal{G}(q,g)| \le g - \Omega(t).$$

Remark 3.2. For comparison, Sapozhenko's original graph container lemma says

(4)
$$\log |\mathcal{G}(q, g)| \le g - \Omega(t/\log d),$$

so the main contribution of Lemma 3.1 is to improve the $\Omega(t/\log d)$ -term in the rhs of (4) to $\Omega(t)$. This improvement plays a crucial role in the current work: the bound in (4) would give a weaker bound, $2^{g-\Omega(N/(\sqrt{d}\log d))}$, in (10).

Proof of Lemma 3.1. Given q and g, Lemma 2.6 gives $\mathcal{W} = \mathcal{W}(q,g)$ at cost $O(t \log^2 d/\sqrt{d})$. So it suffices to show that given $(S,F) \in \mathcal{W}$, the cost of specifying $A \in \Phi^{-1}(S,F)$ is at most $g - \Omega(t)$.

Let $\gamma \in (0,1)$ be a constant TBD. (We don't try to optimize γ .)

Case 1. If $|S| < g - \gamma t$, then we specify A by picking a subset of S, which costs $|S| = g - \gamma t$.

Case 2. If $|S| \ge g - \gamma t$, then we first fix an arbitrary closed $A^* \in \Phi^{-1}(S, F)$ (we can simply pick *any* member of $\Phi^{-1}(S, F)$ and take its closure). Note that this choice is free, and $|A^*| = q$ by the definition of $\mathcal{G}(q, g)$.

The crucial observation is that (letting $G^* = N(A^*)$)

$$(G^* \setminus G, G \setminus G^*)$$
 determines $(G, [A])$.

In what follows we first specify $G^* \setminus G$ and $G \setminus G^*$ from which we have [A], and then specify $A \subseteq [A]$.

We first bound the cost of $G^* \setminus G$. Since $G^* \setminus G \subseteq G^* \setminus F$, the cost of $G^* \setminus G$ is at most (using Lemma 2.6 (b))

(5)
$$|G^* \setminus F| = |G^*| - |F| \le |G| - |S| + O(t/(\sqrt{d}\log d)) \le (1 + o(1))\gamma t.$$

Next, we bound the cost of $G \setminus G^*$. Observe that

$$G \setminus G^* = N([A] \setminus A^*) \setminus G^*,$$

because each $x \in G \setminus G^*$ has a neighbor in [A] and none in A^* . So we may specify $G \setminus G^*$ by specifying a $Y \subseteq [A] \setminus A^* \subseteq S \setminus A^*$ with $G \setminus G^* = N(Y) \setminus G^*$. Moreover, we only need $Y \subseteq S \setminus A^*$ of size at most $|G \setminus G^*| \le g - |F| \le (1 + o(1))\gamma t$, by letting Y contain one neighbor of x for each $x \in G \setminus G^*$.

Now, since (again using Lemma 2.6 (b))

$$|S \setminus A^*| = |S| - q \le |F| + o(t) - q \le g - q + o(t) \le (1 + o(1))t,$$

the cost of specifying Y from $S \setminus A^*$ is at most

(6)
$$\log \binom{(1+o(1))t}{(1+o(1))\gamma t} \le (1+o(1))H(\gamma)t$$

where $H(\cdot)$ is the binary entropy function. Finally, once we have [A], we specify A by picking a subset of [A], which costs

$$(7) q = g - t.$$

Summing up (5), (6), and (7), we bound the total cost for Case 2 by

(8)
$$(1+o(1))\gamma t + (1+o(1))H(\gamma)t + (g-t).$$

Now, choose γ so that (8) is less than (say) g - t/2, and the lemma follows.

4. Proof of Theorem 1.1

We show that the rhs of (1) is an upper bound on $\log \operatorname{bis}(Q_d)$. We first dispose of the minor cost for small independent sets.

Proposition 4.1. There is a constant $\alpha \in (0, 1/2)$ such that

(9)
$$|\{I \in \mathcal{B} : |I| \le \alpha N\}| = 2^{(1 - \Omega(1))N/2}.$$

Proof. The lhs of (9) is at most (with plenty of room)

$$\left[\sum_{0 \le k \le \alpha N/2} \binom{N/2}{k}\right]^2 \le 2^{H(\alpha)N}$$

(the inequality uses Proposition 2.1), and the rhs is less than $2^{(1-\Omega(1))N/2}$ for small enough constant α .

Let $\mathcal{B}' = \{I \in \mathcal{B} : |I| > \alpha N\}$ where α is the constant in Proposition 4.1. A natural way to specify a balanced independent set I is to choose a set $A \subseteq \mathcal{E}$ and a set $B \subseteq \mathcal{O} \setminus N(A)$ so that |A| = |B| (and take $I = A \cup B$). Moreover, since $[I \cap \mathcal{E}]$ and $[I \cap \mathcal{O}]$ have no edges between them, A and B must satisfy $\min\{|[A]|, |[B]|\} \le N/4$ (because $|N(X)| \ge |X|, \ \forall X \subseteq \mathcal{E} \text{ or } \mathcal{O}$). Thus, $|\mathcal{B}'|$ is at most

$$2 \times \sum_{g > \alpha N/2} \sum_{\substack{A \subseteq \mathcal{E}: |N(A)| = g \\ |A| \ge \alpha N/2 \\ |[A]| \le N/4}} |\{B \subseteq \mathcal{O} \setminus N(A)\}|$$

$$= 2^{N/2+1} \sum_{g > \alpha N/2} 2^{-g} |\{A \subseteq \mathcal{E}: |N(A)| = g, |A| \ge \alpha N/2, |[A]| \le N/4\}|.$$

Our main task is to show that

(10) given
$$g = \Omega(N)$$
, $|\{A \subseteq \mathcal{E} : |N(A)| = g, |A| \ge \alpha N/2, |[A]| \le N/4\}| \le 2^{g - \Omega(N/\sqrt{d})}$,

from which it follows that (with Proposition 4.1)

$$|\mathcal{B}| \leq 2^{(1-\Omega(1))N/2} + 2^{N/2+1} \sum_{g > \alpha N/2} 2^{-\Omega(N/\sqrt{d})} = 2^{(1-\Omega(1/\sqrt{d}))N/2}.$$

In the rest of the paper, we show (10). In what follows, we always assume that g and A satisfy the restrictions in (10).

Notation.

Recall that a 2-component of A is a maximal 2-linked subset of A (see Section 2).

- A_i 's: 2-components of A.
- $G_i = N(A_i), G = \bigcup_i G_i = N(A).$
- $g_i = |G_i|$, $a_i = |A_i|$, $q_i = |[A_i]|$, $t_i = g_i q_i$.
- $c(A) = \sum_i q_i$ (note that $|A| \le c(A) \le |[A]|$).
- $g = |G| (= \sum_i g_i)$.
- A_i (or simply i) is $\begin{cases} isolated & \text{if } a_i = 1 \text{ (equiv. } g_i = d); \\ small & \text{if } A_i \text{ is not isolated and } g_i < d^4; \\ large & \text{otherwise.} \end{cases}$

Note that the classification in the above bullet point is entirely determined by g_i .

By Proposition 2.5,

$$\sum \{a_i : i \text{ isolated or small}\} = O(N/d),$$

so in particular, we have (since $|A| = \Omega(N)$)

(11)
$$\sum \{g_i: i \text{ large}\} > \sum \{a_i: i \text{ large}\} = \Omega(N).$$

Proof of (10). Observe that (since $|A| \le c(A) \le |A|$) it suffices to show that given g as in (10) and q with $\alpha N/2 \le q \le \min\{N/4, g\}$, the number of A's in $\mathcal E$ with c(A) = q and |N(A)| = g is at most $2^{g - \Omega(N/\sqrt{d})}$ (since then summing this up over all q's gives (10)).

Given q and g, we first decompose (q,g) into $\{(q_i,g_i)'s\}$ so that $\sum_i q_i = q$ and $\sum_i g_i = g$ (and then specify A_i 's satisfying $|[A_i]| = q_i$ and $|G_i| = g_i$). The number of elements in a decomposition $\{(q_i,g_i)'s\}$ is at most g/d, so Proposition 2.2 bounds the cost of the g_i 's by $(g/d)\log(ed)$ and that of the q_i 's by

$$\begin{cases} (g/d)\log(ed) & \text{if } (g>) \ q>2g/d; \\ 2g/d & \text{if } q\leq 2g/d. \end{cases}$$

Therefore, the total cost of the specification of q_i 's and g_i 's is at most

$$O(g \log d/d).$$

Lemma 4.2. Given (q_i, g_i) , if i is isolated or small, then the cost of A_i with $|[A_i]| = q_i$ and $|G_i| = g_i$ is at most g_i .

Proof. The cost of an isolated i is at most

$$\log(N/2) = d - 1 \le g_i.$$

For a small i, we use Proposition 2.3 to bound the cost of $[A_i]$ by

$$\log(N/2) + O(q_i \log d)$$

 $(\log(N/2))$ is the cost for the "specified vertex" in Proposition 2.3). Once we have $[A_i]$ we specify A_i by choosing each subset of $[A_i]$, which costs q_i . Therefore, the total cost for small i's is

$$\log(N/2) + O(q_i \log d) + q_i \le q_i,$$

where the inequality follows from the fact that $g_i/2 \ge d-1$ and Proposition 2.5.

Finally, given (q_i, g_i) such that i is large, Lemma 3.1 bounds the cost for A_i with $|[A_i]| = q_i$ and $|G_i| = g_i$ by

$$(13) g_i - \Omega(t_i)$$

(here we need the assumption that $(q_i \le)$ $q \le N/4$ to apply Lemma 3.1).

Summing up the costs in (12), Lemma 4.2 and (13), we have the cost for A at most

(14)
$$O(g \log d/d) + g - \sum {\Omega(t_i): i \text{ large}}.$$

Now Proposition 2.4 gives $t_i = \Omega(g_i/\sqrt{d})$ for all i, so by (11) we bound (14) by

$$(15) g - \Omega(N/\sqrt{d}).$$

Acknowledgment. The author is grateful to Matthew Jenssen for insightful conversations and helpful comments on the first draft of this paper.

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