Lecture 1

Motivating examples

- $\mathcal{I}(G) := \{ \underbrace{\text{indept sets}}_{\text{vt set in which no pairs are adj.}} \text{ in } G \}$, $\quad i(G) := |\mathcal{I}(G)|$

E.g. 1. G: d-reg, n vts, bip. (default: d, n large)

$n/2$

$n/2$

$$i(G) \geq 2^{n/2}$$

Prop. $i(G) \leq 2^{(1+o_d(1))n/2}$ $\quad \left(= 2^{n/2} \times 2^{o(n/2)} \right)$.

Pf. entropy, graph container (main focus)

E.g. 2. Q_d: d-dimentional hypercube

e.g. Q_3

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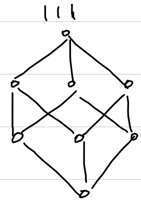

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$V(Q_d) = \{0,1\}^d$

$v \sim w$ iff differ on exactly 1 coord

Q_d: d-reg, bip on $\mathcal{E} \cup \mathcal{O}$ (unique)

$\mathcal{E}$

$\mathcal{O}$

$|V(Q_d)| = 2^d =: N$

$i(G)$?

Thm. (Korshunov - Sapozhenko '83)

$$i(Q_d) \sim 2 \cdot \sqrt{e} \cdot 2^{N/2}$$

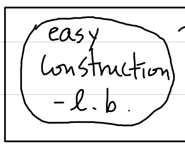
as $d \to \infty$.

$f \sim g \iff f/g \to 1$

i.e., $f = (1+o(1))\, g$.

No closed formula for $i(Q_d)$.

Pf goes by



negli. $\to$ typical structure (e.g. sampling arg.)

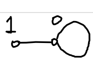
$I(Q_d)$

IS $\xrightarrow{\text{gen.}}$ graph hom's.

Def. $f: V(G) \to V(H)$ is a graph homomorphism if

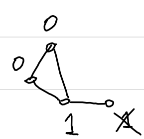
$$\{u,v\} \in E(G) \implies \{f(u), f(v)\} \in E(H).$$

e.g. H:



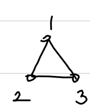
IS

G:

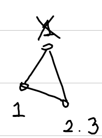


e.g. proper coloring

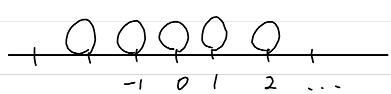
H: K_q



G:



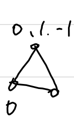
e.g. H:



$\mathbb{Z}$

1- Lipschitz function

G:



Entropy basics (Pfs : EX)

X, Y : discrete rand var's

$p(x) := \mathbb{P}(X = x)$, $\log = \log_2$

Entropy of X : $H(X) = \sum_x p(x) \log \frac{1}{p(x)}$. $\sim$ amount of (randomness / surprise, info) in X

$(0 \cdot \log \frac{1}{0} = 0)$

connection

Properties :

① $H(X) \le \log |\mathrm{Range}(X)|$, $=$ iff X $\underbrace{\text{uniform}}_{\text{"most random"}}$

$(\because)$ $H(X) = \sum_x p(x) \cdot \log \frac{1}{p(x)} \underset{\substack{\uparrow \\ \text{Jensen}}}{\le} \log \left(\sum_x p(x) \cdot \frac{1}{p(x)} \right)$ ∎

② $H(X_1, \cdots, X_k) \le H(X_1) + \cdots + H(X_k)$

Want(?) $H(X, Y) = H(X) + \underbrace{H(Y|X)}_{\text{conditional entropy}}$

Def. T: event,

$$H(X|T) = \sum_x p(x|T) \log \frac{1}{p(x|T)} \qquad p(x|T) = \mathbb{P}(X = x \mid T)$$

Y : r.v.

$$H(X|Y) = \mathbb{E}[H(X|Y=y)]$$

$$= \sum_y p(y) \cdot H(X|Y=y)$$

③ $H(X_1, X_2, \cdots, X_k) = H(X_1) + H(X_2 | X_1) + H(X_3 | X_1, X_2) + \cdots$

④ If Y determines Z, then

$$H(X|Y) \le H(X|Z)$$

Shearer's Lemma gen of ②

e.g. $X = (X_1, X_2, X_3, X_4)$

$$H(X) \le \frac{2}{3} H(X_1, X_2, X_3) + \frac{1}{3}\left[H(X_1, X_4) + H(X_2, X_4) + H(X_3, X_4)\right].$$

Lem $X = (X_1, \cdots, X_k)$ random vector, $\alpha : 2^{[k]} \to \mathbb{R}^{\ge 0}$

If $\sum_{A \ni i} \alpha(A) \ge 1 \quad \forall i \in [k]$, then

$$H(X) \le \sum_{A \subseteq [k]} \alpha(A) \cdot H(\underbrace{X_A}_{= (X_i \,:\, i \in A)}).$$

Prop. G: d-reg, n vts, bip.

$$i(G) \le 2^{(1+O(1/d)) n/2}$$

Starting pt: $f \in \mathcal{I}(G)$ unif rand $\Rightarrow H(f) = \log i(G)$

ETS. $H(f) < (1 + O(\frac{1}{d})) n/2$

$\boxed{\text{ETS. } H(f) < (1+O(\tfrac{1}{d}))\tfrac{n}{2}}$

view $f: V(G) \to \{0,1\}$

notation: $f_x = f(x)$, for $A \subseteq V$, $f_A = (f_x : x \in A)$

$f(A) = \{f_x : x \in A\}$ "palette"

$\in$ $\{0\}$ or $\{1\}$ or $\{0,1\}$

$N_x = N(x)$.

$$H(f) = H(f_Y) + H(f_X \mid f_Y) \qquad (*)$$

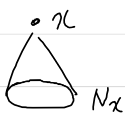
$$H(f_X \mid f_Y) \le \sum_{x \in X} H(f_x \mid f_Y) \le \sum_{x\in X} H(f_x \mid f(N_x))$$

$$H(f_Y) \overset{Sh}{\le} \frac{1}{d}\sum_{x\in X} H(f_{N_x}) = \frac{1}{d}\sum_{x\in X}\left[H(f(N_x)) + H(f_{N_x} \mid f(N_x))\right]$$

$$\to (*) \le \sum_{x\in X} \underbrace{\frac{1}{d}\underbrace{H(f(N_x))}_{O(1)}}_{O(\frac{1}{d})} + \underbrace{\left[H(f_x \mid f(N_x)) + \frac{1}{d} H(f_{N_x} \mid f(N_x))\right]}_{(**)}$$

Claim. ≤ 1 (then done)

picture:



x	0,1	0	0
N_x	0	1	0,1

rigorously:

$$(**) = \sum_{c} \mathbb{P}(f(N_x)=c)\underbrace{\left[H(f_x \mid f(N_x)=c) + \frac{1}{d}H(f_{N_x} \mid f(N_x)=c)\right]}_{\le 1}$$

≤ 1 ∎