Lecture 5

We prove CL 1.

Def. $\mathcal{G}(a,g) := \{A \subseteq \mathcal{O} : A$ 2-linked, $|[A]| = a$, $|N(A)| = g\}$

$t := g - a$

Def. (System of containers)

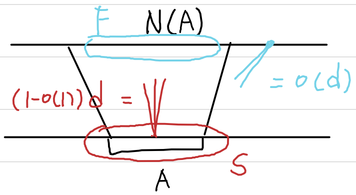
A <u>ψ- approximation</u> for $A \subseteq \mathcal{O}$ is a pair $(\overset{\mathcal{O}\,\cup\!|}{S}, \overset{\subseteq \mathcal{E}}{F})$

s.t.

① $[A] \subseteq S$, $N(A) \supseteq F$

② $\forall u \in S$, $d_F(u) \geq d - \psi$

③ $\forall v \in \mathcal{E} \setminus F$, $d_S(v) \leq \psi$



Think $\psi \approx d^{2/3}$

<u>CL 1.</u> (constructing containers)

There is a family $\mathcal{U} = \mathcal{U}(a,g) \subseteq 2^{\mathcal{O}} \times 2^{\mathcal{E}}$ with

$$|\mathcal{U}| \leq \frac{N}{2} 2^{O(g \log d / \psi)} \quad \text{s.t.}$$

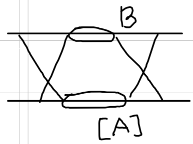
every $A \in \mathcal{G}(a,g)$ has a ψ-approximation (S,F) in $\mathcal{U}$.

Construction in two steps.

$e(A, \overline{N(A)}) = \phi$

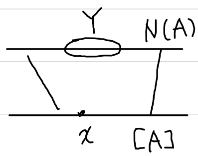
Step 1. crude approx via classic results. ($\neq$ original)

Def. For $A \subseteq \mathcal{O}$, say B <u>covers</u> A if $[A] \subseteq N(B)$ & $B \subseteq N(A)$.



Claim. $\forall A \in \mathcal{G}(a,g)$, $\exists$ a cover B of A of size $O(\underbrace{g\log d/d}_{\text{very small}})$.

pf (?)



Y : rand subset of $N(A)$ including $y \in N(A)$ with prob $\log d/d$ indeptly.

EX. Finish pf.

$\mathcal{B}$ = {covers of $A \in \mathcal{G}(a,g)$ from above}

Check. $B \in \mathcal{B}$. then B is 4-linked ($Q_d^4[B]$ connected).

Claim. $|\mathcal{B}| \le N/2 \cdot 2^{O(g\log^2 d/d)}$.

Recall from p session: G: max deg d. $v \in V(G)$,

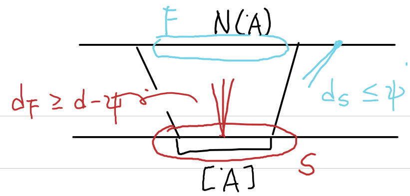
$\rightarrow$ (# $A \subseteq V(G)$ with $v \in A$, $|A| = k$, $G[A]$ connected)

$\le (ed)^k$.

$\Rightarrow |\mathcal{B}| \le N/2 \cdot (ed^4)^{O(g\log d/d)} = N/2 \cdot 2^{O(g\log^2 d/d)}$.

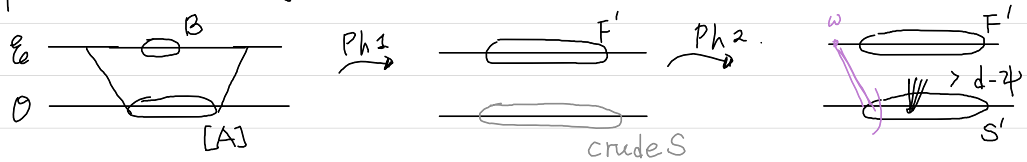
vs. $2^{O(g\log d/\psi)} = 2^{O(g\log d/d^{2/3})}$

Step 2. Refine the crude approx.



<u>Algorithm</u> Fix an ordering $\preceq$ of $V(Q_d)$.

Input : $A \in \mathcal{G}(a,g)$ and its cover B.



Phase 1: Grow F', "achieve" d_F cond.

- Set $F' = B$
- If $\exists\, u \in [A]$ s.t. $\overbrace{d_{N(A)\setminus F'}(u) \geq \psi}^{\text{makes containers "cheap"}}$, then pick the smallest u and update $\underbrace{F'}_{\subseteq N(A),\ \text{always}} \leftarrow F' \cup N(u)$. Repeat until $\nexists$ such u.

Phase 2: Shrink S', "achieve" d_S cond.

- Set $S' = \{ u \in \mathcal{O} : d_{F'}(u) \geq d - \psi \}$. $(\supseteq [A])$
- If $\exists\, w \in \mathcal{E}_e \setminus N(A)$ s.t. $d_{S'}(w) \geq \psi$, then pick the smallest u and update $S' \leftarrow S' \setminus N(w)$. Repeat until $\nexists$ such w.

Output $S = S'$ $(\supseteq [A])$

$F = F' \cup \{ w \in \mathcal{E}_e : d_S(w) \geq \psi \}$. $(\subseteq N(A)$ (why?))

EX. The output (S, F) is a ψ-approx of A.

Claim. # outputs (running for all $A \in \mathcal{G}(a,g)$ s.t. B covers A)

$= 2^{O(g \log d / \psi)}$

$\rightarrow$ total # (S,F) $= \frac{N}{2} \cdot \cancel{2^{O(\log^2 d / d)}} \times 2^{O(g \log d/\psi)}$

$\Rightarrow$ CL 1.

Claim. # outputs (running for all $A \in \mathcal{G}(a,g)$)

$= 2^{O(g \log d / \psi)}$.

(Pf.)

Cost for phase 1: Final F' in Ph 1 is det'd by the u's that we picked.

iterations $\le g/\psi$ (why?)

" # vts we picked ($\subseteq [A] \subseteq N(B)$)

$\rightarrow$ Specify g/ψ vts from $\underbrace{N(B)}_{\text{size } \le d \cdot |B| \le dg}$

$\rightarrow \binom{dg}{g/\psi} = 2^{O(g \log d / \psi)}$.

F' determines "initial" S' name this S_0

Cost for phase 2: Final S' in Ph 2 is det'd by w's

iterations $\le |S_0|/\psi$ (Why?) $\le 2g/\psi$ $|S_0| \lesssim |F'| \le g$

$$\left(\begin{array}{l} |S_0|(d-\psi) \le \nabla(S_0, F') \le d \cdot |F'| \le dg \\ \quad \rightarrow |S_0| \le 2g \end{array} \right)$$

$\rightarrow$ Specify $2g/\psi$ vts from $\underbrace{N(S_0)}_{\text{size } \le d \cdot 2g}$

$\rightarrow \binom{2dg}{2g/\psi} = 2^{O(g \log d / \psi)}$ ∎