Lecture 4

Colorings of Q_d

$\underbrace{\text{Colorings}}_{\text{graph hom's}}$ of $\underbrace{Q_d}_{\text{weak expanders}}$

$\mathcal{Col}_q(G)$: { q-colorings of $V(G)$ }

$C_q(G) := |\mathcal{Col}_q(G)|$.

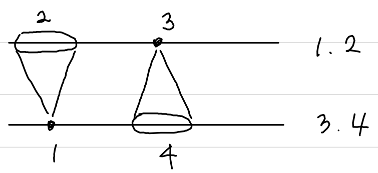
Model thm. (Kahn - P.)

$C_4(Q_d) \sim \underbrace{6 \cdot e \cdot 2^N}_{\text{asymp. l.b.}}$ $\quad (N = |V(Q_d)|)$

cf. $i(Q_d) \sim 2 \cdot \sqrt{e} \cdot 2^{N/2}$

compare

Colors = $\{1, 2, 3, 4\}$



} ground state $\sim$ "GOOD" 4-colorings

4-colorings with k non-nearby flaws

$\approx \binom{N}{k} \cdot 2^{N-dk} \xrightarrow{\text{sum}} e \cdot 2^N$. (Why 6 ?)

Goal. "bad" colorings are negli. $\rightarrow$ will show this for a very special case

cf. Jenssen - Keevash

$$C_6(Q_d) \sim \binom{6}{3} 3^N \cdot e^{\left(\frac{4}{3}\right)^d}.$$

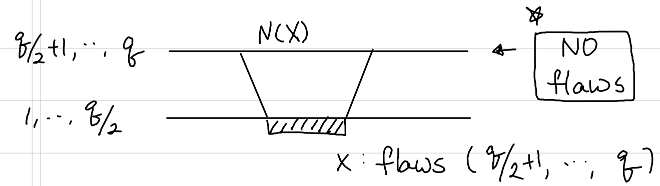
$$C_8(Q_d) \sim \binom{8}{4} \cdot 4^N \cdot e^{\left(\frac{3}{2}\right)^d + \frac{d^2+41d-54}{108}\left(\frac{9}{8}\right)^d}.$$

Behavior changes as q grows.

We stick to $q = O(1)$.

We analyze the following special case:

"Problematic" (revisited) $\quad q$ even



X: 2-linked

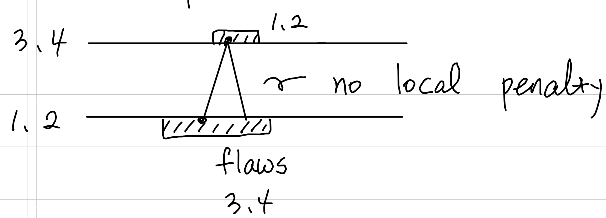
$|[X]| = x$

$|N(X)| \approx (1 + c/\sqrt{d}) \cdot x$

Claim. # "such" q-colorings $\ll (q/2)^N$. $\quad (N = |V(Q_d)|)$

(driving force: penalty from bdry)

Rmk. real problematic situation:



First try. (will fail)

Use Containers $\rightarrow$ # X's $= 2^{|N(X)| - \tilde{\Omega}(|N(X)| - |X|)} \cdot \cancel{\times \frac{N}{2}}$

"such" q-colorings

$$\leq 2^{|N(X)| - \tilde{\Omega}(|N(X)| - |X|)} \cdot \underbrace{(q/2)^{N - |N(X)|} \cdot \left(\frac{q}{2} - 1\right)^{|N(X)|}}_{(q/2)^N \cdot \left(1 - \frac{2}{q}\right)^{|N(X)|}}$$

$$\gg (q/2)^N.$$

Lesson. q larger $\Rightarrow$ penalty smaller

containers too expensive.

Fix :

CL 1 : constructing containers (cheap)

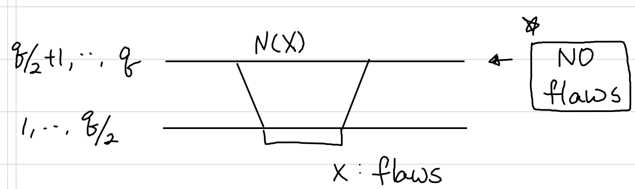
CL 2 : specifying X's from containers (expensive) $\Big\}$ too expensive

then coloring

Intuition: bad colorings cluster.

$\rightarrow$ skip CL2 & specify colorings from containers

Formal setting



X : 2-linked

$|[X]| = a$

$|N(X)| = g \quad (\approx (1 + c/\sqrt{d}) \cdot a)$

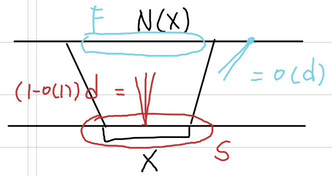
$\mathcal{C} :=$ { q-colorings whose flaws satisfy }

$f \in \mathcal{C}, \quad X_f = \{\text{flaws of } f\}$

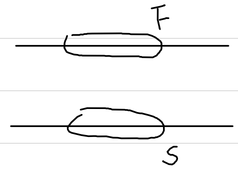
CL 1 $\Rightarrow$ containers (S, F) of $\{X_f : f \in \mathcal{C}\}$ (cheap)

Claim. Given (S, F), the number of f's in $\mathcal{C}$ s.t. (S, F) is a ψ-approx of X_f is $o((q/2)^N)$.

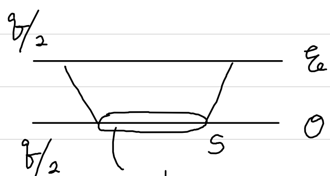
a vt is <u>good</u> if its color follows the ground state.



We only see



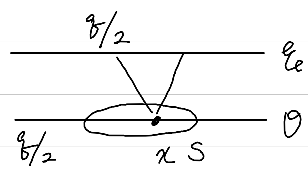
Naive: color S and color the rest?



N.G.

q choices — too expensive

Color $V \setminus S$ and color S?



good? bad?

f_x depends on $f(N_x)$

$\rightarrow$ entropy!

f unif from $\mathcal{C}_e$

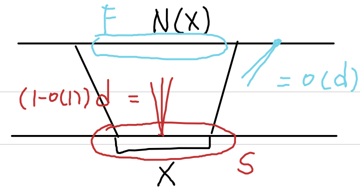
Claim. $H(f) < N \cdot \log(q/2) - $ "large"

$$H(f) = \begin{cases} H(f_{\mathcal{E}_e}) + H(f_{\mathcal{O}} \mid f_{\mathcal{E}_e}) \\ H(f_{\mathcal{O}}) + H(f_{\mathcal{E}_e} \mid f_{\mathcal{O}}) \quad \cancel{\star} \end{cases}$$

($\because$ driving force: penalty from bdry)

$$H(f) = H(f_{\mathcal{O}}) + H(f_{\mathcal{E}_e} \mid f_{\mathcal{O}})$$

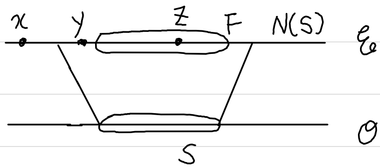
$$\le \sum_{x \in \mathcal{E}_e} \Big[\underbrace{\tfrac{1}{d} \underbrace{H(f_{N_x})}_{\le \sum_{y \in N_x} H(f_y)} + H(f_x \mid f_{N_x})}_{T(x)} \Big] \quad \text{—} \quad \circledast$$



goal.

$< N \cdot \log(q/2)$ − "large"

Recall. f_A vs $f(A)$



① $x \in \mathcal{E}_e \setminus N(S)$: $T(x) \le 2\log(q/2)$ (benchmark)

② $y \in N(S) \setminus F$: N_y can be good or bad → need to reveal $f(N_y)$.

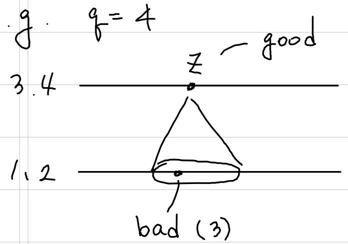
$$T(y) \le \tfrac{1}{d} H(f(N_y)) + \Big[\tfrac{1}{d} H(f_{N_y} \mid f(N_y)) + H(f_y \mid f(N_y))\Big]$$

$$\le \underbrace{O_q(1/d)}_{\text{overpay}} + 2\log(q/2).$$

overpay – will be dominated by "penalty" from ③

③ $z \in F$: $T(z) \le \underbrace{\tfrac{1}{d} H(f(N_z))}_{O_q(1/d)} + \Big[\underbrace{\tfrac{1}{d} H(f_{N_z} \mid f(N_z)) + H(f_z \mid f(N_z))}_{\text{Claim. } \le 2\log(q/2) - \Omega_q(1)}\Big]$

e.g. $q = 4$



$f(z)$	1 color	3	2 $(= q/2)$
$f(N_z)$	3 colors	1	2 $(= q/2)$
$T(z) \le$	$\log 3$	$\log 3$	imposs. (why?)

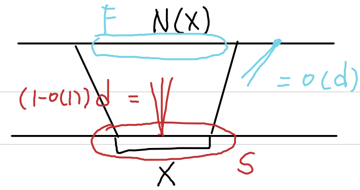
benchmark $T(z) = 2$.

Want $T(z) \le 2 - \Omega(1)$

$$H(f) = H(f_{\mathcal{O}}) + H(f_{\mathcal{E}} \mid f_{\mathcal{O}})$$

$$\le \sum_{x \in \mathcal{E}} \underbrace{\left[\frac{1}{d} \underline{H(f_{N_x})} + H(f_x \mid f_{N_x}) \right]}_{T(x)} \quad — \quad \circledast$$

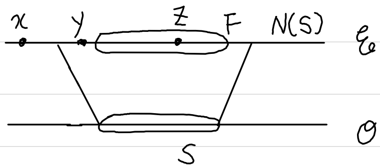
$$\le \sum_{y \in N_x} H(f_y)$$



goal.

$< \quad N \cdot \log(q/2) - $ "large"

Recall. f_A vs $f(A)$



$$\circledast \le N \log(q/2) + O_q(1/d) \cdot |N(S)| - \Omega_q(1) \cdot |F| .$$

Done if $\quad |N(S)| \ll d \cdot |F|$. (why?)

By def of ψ-approx,

① $|F| \gtrsim |S|$

② $|N(S)| \lesssim |F| + o(d) \cdot |S| \lesssim |F| \cdot o(d) \ll d \cdot |F| \quad \blacksquare$

Again, the real issue is ...