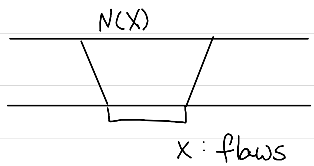
Lecture 3 • Clarify $N(X)$

Recall. • $A \subseteq \mathcal{O}$ is <u>2-linked</u> if $Q_d^2[A]$ is connected

• $[A] := \{v \in \mathcal{O} : N(v) \in N(A)\}$ "closure of A"

Target $i(Q_d) \sim \underbrace{2 \cdot \sqrt{e} \cdot 2^{N/2}}_{\text{lower bd from "GOOD" IS}}$

"Problematic":



X: 2-linked
$|[X]| = x$
$|N(X)| \approx (1 + c/\sqrt{d}) \cdot x$

$\#$ "such" IS $\approx \underbrace{(\#\text{ choices for } X)}_{\text{"cost"}} \cdot 2^{N/2 - \underbrace{|N(X)|}_{\text{"penalty"}}}$ goal $\ll 2^{N/2}$

Need. Cost $\ll 2^{|N(X)|}$

"basic" containers not enough

$i(G) \le |\mathcal{C}| \cdot \underbrace{2^{\max\{|C| : C \in \mathcal{C}\}}}_{\text{main term, loose}}$ $\circledast$

Lem. Cost $< 2^{|N(X)| - \tilde{\Omega}(\underbrace{|N(X)| - |X|}_{\gtrsim |N(X)|/\sqrt{d}})} \cdot \underbrace{\frac{N}{2}}_{\text{cost for } v_0 \in X \text{ negli.}}$. (then done)

Main Point. Improve $\circledast$

Def. $\mathcal{G}(a,g) := \{ A \subseteq \mathcal{O} : A \text{ 2-linked}, \ |[A]| = a, \ |N(A)| = g \}$

$t := g - a$

improvable

<u>Lem</u>* $\quad |\mathcal{G}(a,g)| \le 2^{g \boxed{- \Omega(t/\log^2 d)}} \cdot \frac{N}{2}$

pt of arg, expansion crucial.

A system of containers for $\mathcal{G}(a,g)$

Note. $\underbrace{A \cup (\mathcal{E}_e \setminus N(A))}_{IS} \subseteq S \cup (\mathcal{E}_e \setminus F)$

But, the pf doesn't require bipartiteness.

it requires "expansion" & "regularity"

quantification of $o(d) \to \underset{\text{think}}{\psi \approx d^{2/3}}$ (just to be concrete)

Def. (System of containers)

A ψ-approximation for $A \subseteq \mathcal{O}$ is a pair (S, F) $\quad \overset{\mathcal{O}}{\underset{\mathrm{U}}{S}}, \ F \subseteq \mathcal{E}_0$

s.t.

① $[A] \subseteq S$, $\quad N(A) \supseteq F$

② $\forall u \in S$, $\quad d_F(u) \geq d - \psi$

③ $\forall v \in \mathcal{E}_0 \setminus F$, $\quad d_S(v) \leq \psi$

container lemma

CL 1. (constructing containers)

There is a family $\mathcal{U} = \mathcal{U}(a, g) \subseteq 2^{\mathcal{O}} \times 2^{\mathcal{E}_0}$ with

$$|\mathcal{U}| \leq \frac{N}{2} 2^{O(g \log d / \psi)} \qquad \text{s.t.}$$

every $A \in \mathcal{G}(a, g)$ has a ψ-approximation (S, F) in $\mathcal{U}$.

For experts : some hidden terms

Notn : $A \sim (S, F)$ if (S, F) is a ψ-approx of A.

CL 2 (spec. A from containers)

For each $(S, F) \in \mathcal{U}$, there are at most

$$2^{g - \Omega(t / \log^2 d)}$$

$A \in \mathcal{G}(a, g)$ s.t. (S, F) is a ψ-approx of A.

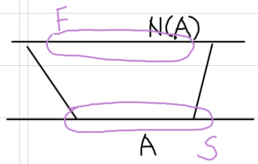
CL 1 + CL 2 $\Rightarrow$ Lem*

$$\left(\because \ 2^{O(g \log d / \psi)} \leq 2^{O(t \log d / d^{1/6})} \ll 2^{\Omega(t / \log^2 d)} \right)$$

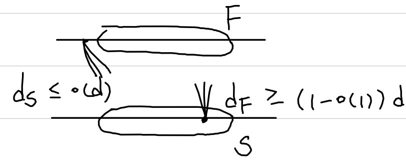
$t \gtrsim g / \sqrt{d}$

First focus on CL 2.

Pf of CL 2 assuminging CL 1.



We see



Goal. # A's $\bar{w}$ $A \sim (S, F)$ $\le 2^{|N(A)| - \Omega(t/\log^2 d)}$.

Let (S, F) be given.

Immediate case: $|S| \le |N(A)| - ct/\log^2 d$ (why?)

$\to$ WMA $|S| > |N(A)| - ct/\log^2 d$

Plan ① Using "regularity", show that
$|F| \gtrsim |S|$

$\to$ $|N(A) \setminus F|$ "small" $\to$ "cheap".

$\to$ determines $N(A)$ $\to$ determines $[A]$.

② Specify A as a subset of $[A]$

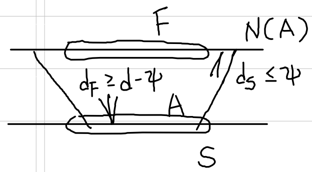
$\to$ cost $= 2^{|[A]|} = 2^{|N(A)| - t}$!!

big enough to kill the cost for ①.

Note. ① uses regularity, ② uses expansion.

Implementation of Plan

Prop. $|F| \geq |S| - O\left(\frac{t\psi}{d}\right)$. (EX)



$\Bigg($ Naive : $|S|(d-\psi) \leq \nabla(S, F) \leq d|F| \Rightarrow |F| \geq |S| - \frac{|S|\psi}{d}$ $\ddot{\frown}$

$\to$ Should use def of ψ-approx more intensively. $\Bigg)$

Then, $|N(A) \setminus F| = |N(A)| - |F|$

$$\leq |N(A)| - \underbrace{|S|}_{|N(A)| - ct/\log^2 d} + O\left(\frac{t\psi}{d}\right) = O\left(t/\log^2 d\right).$$

Specify $N(A) \setminus F$ <u>as a subset of $N(S)$</u>

cf. # of k-sets in $[n]$ $= \binom{n}{k} \approx 2^{O(k \log(\frac{n}{k}))}$

$$\to \# \leq \binom{\overset{\subseteq N^2(F)}{|N(S)|}}{O(t/\log^2 d)} \leq \binom{d^2 g}{O(t/\log^2 d)} \underset{\substack{\uparrow \\ t \gtrsim g/\sqrt{d}}}{=} 2^{O(t/\log d)}.$$

$\blacksquare$