COUNTING QUADRATIC POINTS ON DEL PEZZO SURFACES

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Introduction. During our two weeks at the Institute for Advanced Study, we studied the Manin–Peyre conjecture on symmetric squares $\mathrm{Sym}^2 V$ of generalized del Pezzo surfaces V. The only surfaces for which this problem has been studied in the past are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by Le Rudulier [LR19]. We are interested in generalizing this result to a larger class of del Pezzo surfaces.

Recall that the Manin–Peyre conjecture states that:

Conjecture 1. *Let k be a number field and V a smooth Fano variety such that $V(k)$ is Zariski dense. Let H be an anticanonical height function. Then, there exists a cothin subset $\mathcal{U} \subset V(k)$, such that*

$$N(\mathcal{U}, B) := \#\{x \in \mathcal{U} : H(x) \leq B\} \sim c_{Peyre,k} B(\log B)^{\rho(V)-1}, \qquad (0.1)$$

where $\rho(V) = \mathrm{Rank}\,\mathrm{Pic}\,V$.

Peyre [Pey95], relying on his own previous work and also on work from Batyrev–Tschinkel, conjectured the value of the leading constant and that it should take the form $c_{\mathrm{Peyre},k} = \alpha \times \beta \times \tau$, where

- α measures the size of the cone of effective divisors of V,
- $\beta = \#\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/k), \mathrm{Pic}\, V_{\overline{\mathbb{Q}}})$ and
- τ is a Tamagawa number associated to the variety V.

Goal. Prove the Manin–Peyre conjecture for $\mathrm{Sym}^2 V$ where V is a del Pezzo surface. That is, identify the correct cothin set $\mathcal{V} \subseteq \mathrm{Sym}^2 V(\mathbb{Q})$ for which $N(\mathcal{V}, B)$ has the predicted order of growth and leading constant.

General approach. We roughly follow the approach of Le Rudulier [LR19]: that is, we first count *pure* K-points of bounded height

$$N^*(\mathcal{U}_K, B) = \#\{x \in \mathcal{U}_K : H(x) \leq B, \kappa(x) = K\}$$

over all number fields K with $[K : \mathbb{Q}] = 2$ for a uniform family of cothin sets $\mathcal{U}_K \subseteq V(K)$. Next we relate $N(\mathcal{V}, B)$ to the sum $\sum_K N(\mathcal{U}_K, B)$ over all quadratic extensions. We would then relate the sum to the values predicted by the Manin–Peyre conjecture.

We have completed the ideas we had before our visit and generalized them where necessary to obtain the following results:

(1) $\beta(V_K) = \beta(\mathrm{Sym}^2 V)$ and $\rho(V_K) = \rho(V)$ for all but finitely many number fields $K/\mathbb{Q}$.
(2) Hence, to obtain the correct order of growth one may need to exclude finitely many terms in the sum $\sum_K N(\mathcal{U}_K, B)$.
(3) We formulated an expression for $\tau(\mathrm{Sym}^2 V)$ in terms of the local densities of V.

(4) We proved general results concerning sums of Euler products over quadratic number fields, which vastly generalize the work of Schmidt [Sch95] and Le Rudulier [LR19]. For this particular project the relevant result is

$$\sum_{|\Delta|\leq Y} \tau(V_{K_\Delta}) = \frac{1}{2}\tau(\operatorname{Sym}^2 V)\log Y + O(1).$$

(5) The logarithm component agrees with Manin's conjecture since $\operatorname{rk}\operatorname{Pic}(\operatorname{Hilb}^2 X) = \operatorname{rk}\operatorname{Pic} X + 1$.

During our visit at the IAS we worked on the following main problems. The first two points are necessary to prove the Manin–Peyre conjecture, and the last two are used to compare the leading constant to the one predicted by Peyre.

(1) The computation of $N(\mathcal{U}_K, B)$ turns out to be difficult since the only method of dealing with the condition of $\kappa(P) = K$ in the literature uses the geometry of numbers; this does not apply to general rational surfaces.
(2) The bound on the discriminant of the quadratic extensions K so that if Δ_K is large enough, then $N(\mathcal{U}_K, B) = 0$. In the case of projective spaces, the work of Silverman [Sil84] provides this bound. We showed that such a bound has to be quite precise in general; it is impossible to sum over the quadratic fields otherwise as the error term from item (1) will overwhelm the main term.
(3) The computation of $\alpha(\operatorname{Sym}^2 V)$ and of the Tamagawa numbers for $\operatorname{Sym}^2 V$.
(4) The constant we obtain is a sum of Peyre's constants which are each an Euler product. We then need to show that this sum yields the Euler product corresponding to Peyre's constant for $\operatorname{Sym}^2 V$.

Our long-term goal is to address each of these questions in general, and we did have some success in this direction. Most notably, however, we were able to completely solve all of them for an infinite family of del Pezzo surfaces. This is the first instance in the literature where the Manin–Peyre conjecture holds over the second symmetric power of a family of surfaces to the best of the authors' knowledge. While this example is interesting in its own right, we also believe that this will serve as an instructive prototype in the future in studying the asymptotic distribution of rational points on $\operatorname{Sym}^2 V$ for V any del Pezzo surface.

An infinite family of examples. Here is the main theorem we obtained during our visit at the IAS. We are currently in the process of writing it up and checking it carefully.

Theorem 2 (Destagnol, Lyczak, Park, Rome (2023))**.** *The Manin–Peyre conjecture holds for the symmetric square of non-split quadric surfaces.*

Our proof goes through addressing all of the problems listed above. Firstly, let us record that for such surfaces we have $\alpha(V) = \frac{1}{2}$ and we can explicitly compute $\tau(V)$. More specifically, we obtained the following results to the first two points in the list above.

(1) For any cothin set the main terms $N(\mathcal{U}_K, B)$ should not differ from the main term of $N(V(K), B)$, and we have, for example, the following result that can be easily generalized to arbitrary number fields.

Proposition 3.

$$N(V(\mathbb{Q}), B) = 4\prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-2}\left(1 - \frac{1}{p^2}\right)\left(1 - \frac{\chi(p)}{p^2}\right) B + O(B^{3/4})$$

where χ is the quadratic character associated to the splitting field of V.

To do the count for $N^*(V(\mathbb{Q}), B)$, we use anticanonical heights induced from an adelic height on $\mathbb{P}^1$. We can then apply results on counting pure K-points on projective spaces.

(2) It remains to deal with the sum of the error terms over quadratic extensions. In place of the Silverman bound, we proved the following statement.

Proposition 4.

$$\sup\{s \in \mathbb{R}_{\geq 0} \colon \forall K, P \in V(K) \setminus V(\mathbb{Q}) \Rightarrow H(P) \gg |\Delta_K|^s\} = \frac{1}{2}.$$

This constant, a generalization of Silverman's bound on heights of projective points over number fields, was crucial in proving the theorem above. Using this, we sum the error terms over quadratic fields. Here, for notational simplicity, we have restricted to the quadric surface split by $\mathbb{Q}(i)$, which can be described by the equation $x^2+y^2 = zw$.

Theorem 5. *We have $N(V(K), B) = c_K B + O\left(h_{K(i)} R_{K(i)} |\Delta_{K/\mathbb{Q}}|^{-3/2} B^{7/8}\right)$ where c_K can be explicitly computed. Furthermore,*

$$\sum_{|\Delta_{K/\mathbb{Q}}| \ll B^{1/4}} h_{K(i)} R_{K(i)} |\Delta_{K/\mathbb{Q}}|^{-3/2} \ll B^{1/8}$$

This shows that the error terms has order B and hence does not contribute to the main term which has order $B \log B$, so we are now free to use our prior results to prove the Manin–Peyre conjecture for $\operatorname{Sym}^2 V$.

To compare the constant to the one predicted by Peyre we needed to compute the effective cone of non-split quadrics, which we were able to do. There still remains some work in precisely relating the Tamagawa numbers of V and $\operatorname{Sym}^2 V$, but we believe that this difficulty is surmountable.

Outlook. We will spend the coming months in polishing our results and writing them up for publication. We will also start new projects in the direction of generalizing this result for other del Pezzo surfaces.

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