Statistical physics basics

1. (optional but useful) Write code (in Python, Mathematica, ...) to simulate a statistical physics model on a given input graph with given parameters.

2. Let K_d be the complete graph (clique) on d vertices.

   (a) Compute the hard-core partition function $Z_{K_d}(\lambda)$.

   (b) For $u, v \in K_d$ compute the truncated two-point correlation function.

3. Let $G = G_1 \cup G_2$, the disjoint union of two graphs G_1, G_2. Prove that

$$Z_G(\lambda) = Z_{G_1}(\lambda) Z_{G_2}(\lambda)\,.$$

4. Let $K_{d,d}$ be the complete d-regular bipartite graph (two sets of d vertices L, R) with all d^2 edges between L and R present and no others).

   (a) Compute $Z_{K_{d,d}}(\lambda)$.

   (b) Compute $\mathbb{E}_{K_{d,d},\lambda}|I|$, the expected size of an independent set I drawn from the hard-core model on $K_{d,d}$ at fugacity λ.

5. Prove that the following probability distribution on independent sets of G is the hard-core model on G at fugacity λ. Pick a subset $S \subseteq V(G)$ by including each vertex independently with probability $\frac{\lambda}{1+\lambda}$ and condition on the event that S is an independent set.

6. Prove that the occupancy fraction of the hard-core model on any non-empty graph G is a strictly increasing function of λ.

7. Prove that for the hard-core model on $\mathbb{Z}$, the truncated two-point correlation function decays exponentially fast in the distance, for any $\lambda \geq 0$.

8. Consider a tree T and a vertex $v \in V(T)$ with neighbors $u_1, \ldots, u_d$. Write a formula for the marginal $\mu_{v,\lambda}$ of the hard-core model at fugacity λ in terms of λ and the marginals $\mu^{-v}_{u_1,\lambda}, \ldots, \mu^{-v}_{u_d,\lambda}$ of the neighbors of v in the graph $T \setminus v$.

9. Let $\mathcal{G}_n$ be the set of all (labeled) graphs on n vertices. For $m \in (0, \binom{n}{2})$, determine the maximum entropy probability distribution on $\mathcal{G}_n$ with mean number of edges m. (Recall that the entropy of a probability distribution μ on a finite set Ω is $H(\mu) = -\sum_{x\in\Omega} \mu(x) \log \mu(x)$ with the convention that $0 \log 0 = 0$).

10. Consider the hard-core model on a graph G and let F be the set of vertices that are not in the independent set and have no neighbor in the independent set (they are free to be added to the independent set). Calculate $\mathbb{E}[|F|]$ in terms of derivatives of $\log Z_G(\lambda)$.