1 Lecture 4 exercise

Problem 1. Consider the Pauli matrices:
$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For every Hermitian operator H, define the Pauli Fourier decomposition of ρ as

$$H = \sum_{P \in \{\mathbb{I}, X, Y, Z\}^{\otimes n}} \alpha_P P,$$

where $\alpha_H = \text{Tr}(HP)$.

1. Prove n-qubit Pauli matrices $\{\mathbb{I}, X, Y, Z\}^{\otimes n}$ form an orthonormal basis for $\mathbb{C}^{n \times n}$. In particular show that $\text{Tr}(P \cdot P') = 2^n$ if $P = P'$ and 0 otherwise, where $P, P' \in \{\mathbb{I}, X, Y, Z\}^{\otimes n}$.
2. Prove that the decomposition above is *unique*
3. We will denote $\mathbb{I} = W_{00}, X = W_{01}, Z = W_{10}, Y = W_{11}$ like we mentioned in the lecture. Show that $|\Phi^+\rangle\langle\Phi^+| = \frac{1}{4}\sum_{y \in \mathbb{F}_2^2} W_y (-1)^{y_1 \cdot y_2}$.

Hint: All of them requiring writing the equalities and checking they are true.

Problem 2. Prove that the output distribution of the Bell sampling (on input $|\psi\rangle^{\otimes 4}$) is given by

$$q_\psi(z) = \sum_{a \in \mathbb{F}_2^{2n}} p_\psi(a) p_\psi(z + a), \tag{1}$$

where $p_\psi(a) = \langle\psi|W_a|\psi\rangle^2/2^n$. For $x = (x^1, x^2), y = (y^1, y^2) \in \mathbb{F}_2^{2n}$, define $[x, y] = \langle x^1, y^2\rangle \oplus \langle x^2, y^1\rangle$.

1. For $b \in \mathbb{F}_2^2$, show that $|W_b\rangle\langle W_b| = \frac{1}{4}\sum_{y \in \mathbb{F}_2^2} W_y^{\otimes 2}(-1)^{y_1 y_2 + [b,y]}$, where $|W_b\rangle = (W_b \otimes \mathbb{I})|\Phi^+\rangle$.
2. Show that $\sum_{b \in \mathbb{F}_2^2} |W_b\rangle\langle W_b|^{\otimes 2} = \frac{1}{4}\sum_{b \in \mathbb{F}_2^2} W_b^{\otimes 4}$
3. Let us assume the following two equations

$$\Pi_z = \sum_{x \in \mathbb{F}_2^{2n}} |W_x\rangle\langle W_x| \otimes |W_{x+z}\rangle\langle W_{x+z}| = \frac{1}{4^n} \sum_{x \in \mathbb{F}_2^{2n}} W_x^{\otimes 4} (-1)^{[z,x]} \tag{2}$$

$$q_\psi(z) = \text{Tr}[\Pi_z |\psi\rangle\langle\psi|^{\otimes 4}]. \tag{3}$$

   Now prove Eq. (1) using Eq. (2,3).

Hint: For (i) write down the LHS and RHS and observe they are the same (use Problem 1 (iii) here. For (ii) write down the tensor product of (1.) and expand it out. For (iii) using the definition of $p_\psi(a)$ will prove Eq. (1).

Problem 3. In this problem you will learning parities in the QSQ model. Recall that the class of parities is given by $\mathscr{C} = \{c_S(x) = S \cdot x = \sum_i S_i x_i \pmod 2\}_{S \in \{0,1\}^n}$. Also recall that for a Boolean function $f : \{0,1\}^n \to \{0,1\}$, the Fourier coefficients are given by

$$\widehat{f}(S) = \frac{1}{2^n} \sum_x f(x)(-1)^{S \cdot x}.$$

1. Given a quantum example $|\psi_f\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x, f(x)\rangle$, show that there is an M and τ that satisfies the following: on input $i \in [n]$, the Qstat query returns a τ-approximation of $\sum_{S : S_i = 1} \widehat{f}(S)^2$.

2. Show that there is a polynomial-time implementable M_i and constant τ_i such that by making n many Qstat queries with (M_i, τ_i) one can learn parities

Hint: (i) Apply the Hadamard transform on $|\psi_f\rangle$, construct an M that allows to sum Fourier coefficients over $S : S_i = 1$, (iii) Observe that parities have a single Fourier coefficient, so by choosing $\tau_i = 1/10$ and M_i as in (ii) we can learn the unknown parity S.

Problem 4. In this exercise you will be show that $O(n^2)$ copies of quadratic phase states $|\psi_f\rangle = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle$ where f is degree-2 suffices to learn f using *single-qubit measurements*.

Hint: (i) Measure all but $(n-1)$ qubits in $|\psi_f\rangle$, what information about f can we infer from this? (ii) Repeating (i) $O(n)$ many times to learn a directional derivative (iii) Learn the n directions to learn f.