1 Lecture 3 exercise

Problem 1. Consider the class of quantum states $\frac{1}{\sqrt{2^n}}\sum_x(-1)^{f(x)}|x\rangle$ where $f:\{0,1\}^n \to \{0,1\}$ is a degree-d function over $\mathbb{F}_2$. Show that using $O(n^d \cdot 2^{2d})$ copies of an unknown $\frac{1}{\sqrt{2^n}}\sum_x(-1)^{g(x)}|x\rangle$ where g is degree-d function, one can learn g?

Hint: Use the Schwartz-Zippel lemma that says that for degree-d functions f, g, $\Pr_x[f(x) \neq g(x)] \geq 2^{-d}$ and then use shadow tomography.

Problem 2. In this exercise we first recall the VC-dimension for $\mathscr{C} \subseteq \{c : \{0,1\}^n \to \{0,1\}\}$. For a concept class $\mathscr{C} \subseteq \{c : \{0,1\}^n \to \{0,1\}\}$, write down the matrix $M \in \mathbb{F}_2^{|\mathscr{C}|\times 2^n}$ as $M(c,x) = c(x)$.

The VC dimension is the largest d such that, there exists columns $s_1, \ldots, s_d \subseteq \{0,1\}^n$ s.t. for every $B \subseteq [d]$, there exists $c \in \mathscr{C}$ such that: If $i \in B$, $c(s_i) = 1$, If $i \notin B$, $c(s_i) = 0$

Compute the VC dimension of the concept classes $\mathscr{C} \subseteq \{c : \{0,1\}^2 \to \{0,1\}\}$

Concepts	Truth table			
c_1	0	1	0	1
c_2	0	1	1	0
c_3	1	0	0	1
c_4	1	0	1	0
c_5	1	1	0	1
c_6	0	1	1	1
c_7	0	0	1	1
c_8	0	1	0	0
c_9	1	1	1	1

Concepts	Truth table			
c_1	0	1	1	0
c_2	1	0	0	1
c_3	0	0	0	0
c_4	1	1	0	1
c_5	1	0	1	0
c_6	0	1	1	1
c_7	0	0	1	1
c_8	0	1	0	1
c_9	0	1	0	0

Hint: For $\mathscr{C}_1$, consider the columns $s_1 = 1, s_2 = 3$ and for $\mathscr{C}_2$ consider the columns $s_1 = 2, s_2 = 3, s_3 = 4$

Problem 3. We define the fat-shattering dimension for $\mathscr{C} \subseteq \{c : \{0,1\}^n \to [0,1]\}$. Write down the matrix $M \in [0,1]^{|\mathscr{C}|\times 2^n}$ as $M(c,x) = c(x)$. The γ-fat shattering dimension of $\mathscr{C}$ is the largest d such that, there exists constants $\alpha_1, \ldots, \alpha_d$ and columns $s_1, \ldots, s_d \subseteq \{0,1\}^n$ satisfying the following:for every $B \subseteq [d]$, there exists $c \in \mathscr{C}$ such that: If $i \in B$, $c(s_i) \geq \alpha_i + \gamma$ If $i \notin B$, $c(s_i) \leq \alpha_i - \gamma$.

Let's consider a simple example $\mathscr{C} = \{c : \{0,1\}^2 \to [0,1]\}$ Consider the rows $s_1 = 1, s_2 = 3$. Let

Concepts	Truth table			
c_1	0.9	0.7	0.02	1
c_2	0.88	0.48	0.92	0
c_3	0.1	0.33	0.98	0.22
c_4	0.1	0.55	0.85	0.49
c_5	0.09	0.58	0.1	0.34

$\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\gamma = 0.1$. Now based on this, lets label entries in

- $M(s_1, x) \geq \alpha_1 + \gamma$ as 1 and $M(s_1, x) \leq \alpha_1 - \gamma$ as 0
- $M(s_2, x) \geq \alpha_2 + \gamma$ as 1 and $M(s_2, x) \leq \alpha_2 - \gamma$ as 0

to get Now observe that the entries $\{00, 01, 10, 11\}$ appear in the columns $(1,3)$. So the 0.1-fat

Concepts	Truth table			
c_1	1	0.7	0	1
c_2	1	0.48	1	0
c_3	0	0.33	1	0.22
c_4	0	0.55	1	0.49
c_5	0	0.58	0	0.34

shattering dimension is ≥ 2.

Compute the γ-fat shattering dimension of the concept classes $\mathscr{C} \subseteq \{c : \{0,1\}^2 \to [0,1]\}$

Concepts	Truth table			
c_1	0.02	0.85	0.11	0.57
c_2	0.87	0.9	0.84	0
c_3	0.92	0.39	0.18	0.43
c_4	0.91	0.44	0.81	0.63
c_5	0.84	0.92	0.07	0.88
c_6	0.1	0.77	0.99	0.5
c_7	0.14	0.42	0.95	0.33
c_8	0.2	0.52	0.21	0.47

Concepts	Truth table			
c_1	0.66	0.88	0.86	0.1
c_2	0.57	0.03	0.05	0.92
c_3	0.92	0.11	0.09	0
c_4	0.02	0.98	0.11	0.87
c_5	0.88	0.18	0.96	0.08
c_6	0.5	0.93	0.94	0.98
c_7	0.64	0.01	0.89	0.85
c_8	0.01	0.91	0.13	0.88

Hint: For $\mathscr{C}_1$, consider the columns $s_1 = 1, s_2 = 3$ and for $\mathscr{C}_2$ consider the columns $s_1 = 2, s_2 = 3, s_3 = 4$

Problem 4. Prove that (*i*) γ-fat shattering dim. reduces to VC dim. for a suitable choice of $\alpha_1, \ldots, \alpha_d, \gamma$, (*ii*) VC dimension and γ-fat shattering dim. of $\mathscr{C}$ is at most $\log |\mathscr{C}|$

Hint: Observe that the difference between fat-shattering dimension and VC dimension is $\{\alpha_i - \gamma, \alpha_i + \gamma\} = \{0, 1\}$.

Problem 5. In this exercise you will be showing that the γ-fat-shattering dimension of the class of all n-qubit quantum states is at most $O(n/\gamma^2)$. In order to prove this we will use the following well-known theorem about quantum random access codes.

Theorem 1. *Let $k > n$. For all $y \in \{0,1\}^k$, let ρ_y be an n-qubit mixed state that "encodes" y. Suppose there exist two-outcome measurements $E_1, \ldots, E_k$ such that for all $y \in \{0,1\}^k$ and $i \in [k]$, we have that (i) if $y_i = 0$ then $\mathrm{Tr}(E_i \rho_y) \leq 1/3$, (ii) if $y_i = 1$ then $\mathrm{Tr}(E_i \rho_y) \geq 2/3$. Then $n \geq k/5$.*

You will need to show the following:

1. Let $k, n, \{\rho_y\}$ be as in the theorem above. Suppose there exists two-outcome measurements $E_1, \ldots, E_k$ and $\{\alpha_1, \ldots, \alpha_k\}$ such that for all $y \in \{0,1\}^k$ and $i \in [k]$, we have that (*i*) if $y_i = 0$ then $\mathrm{Tr}(E_i \rho_y) \leq \alpha_i - \gamma$, (*ii*) if $y_i = 1$ then $\mathrm{Tr}(E_i \rho_y) \geq \alpha_i + \gamma$. Then $n \geq k \cdot \gamma^2$.
2. Use the theorem above to conclude that the γ-fat shattering dimension is at most n/γ^2

Hint: In order to prove (1.) how does one amplify the $(\alpha_i + \gamma, \alpha_i - \gamma)$ to $(1/2, 2/3)$? Use amplification by taking $1/\gamma^2$ copies of ρ and then using Chernoff bound, in order to invoke Theorem 1. Once we have (1.) observe that by definition we have obtained a shattered set.