PCMI: RAMSEY THEORY ON GRAPHS - DAY 1

July 2025

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(1) Show that $R(k) \geqslant (1+o(1))e^{-1}k2^{k/2}$, by modifying a random colouring of the complete graph.

(2) Let $R_\ell(k)$ be the ℓ-colour Ramsey number: the minimum n such that every ℓ-colouring of K_n contains a monochromatic K_k. Show that
   (a) $R_\ell(k) \leqslant \ell^{\ell k}$
   (b) By considering a random colouring, show that $R_\ell(k) \geqslant \ell^{k/2}$.
   (c) Show that $R_{s+t}(k) \geqslant (R_s(k)-1)(R_t(k)-1)+1$ and use this show that for all $\ell, k \geqslant 2$ we have
$$R_\ell(k) > 3^{\ell k/6}$$
   whenever $\ell \equiv 0 \mod 3$. What can we say about $\ell \not\equiv 0 \mod 3$?

(3) Show that $R_r(3) \leqslant (r+1)!$.

(4) Let $\chi : \mathbb{N} \to [k]$. Show there is $x, y, z \in \mathbb{N}$ such that $\chi(x) = \chi(y) = \chi(z)$ and $x + y = z$.

(5) Let $n \geqslant k2^{k+2}$. Show that every 2-colouring of the edges of the complete bipartite graph $K_{n,n}$ contains a monochromatic $K_{k,k}$. For which n does there exist a colouring of K_n with no monochromatic $K_{k,k}$?

(6) Define $R^{(r)}(k)$ to be the smallest n so that every colouring of the $[n]^{(r)}$ contains $X \subset [n]$ with $|X| = k$ so that $X^{(r)}$ is monochromatic. Note that $R^{(2)}(k) = R(k)$. Show that $R^{(3)}(k)$ exists. Can you extend to larger $R^{(r)}(k)$?

(7) Define the Ramsey number of H by
$$r(H) = \min\big\{n : \text{ every red/blue colouring } K_n \text{ contains a mono copy of } H\big\}.$$
Let H be a graph and let $k = \chi(H)$ be its chromatic number. Show that $r(H) \geqslant 2^{k/2}$.