PCMI: RAMSEY THEORY ON GRAPHS - DAY 3

July 2025

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(1) Let G be a triangle-free graph on n vertices with $\alpha(G) \leqslant C\sqrt{n \log n}$. Show that $e(G) \geqslant cn^{3/2}\sqrt{\log n}$, for some $c > 0$.

(2) Complete the proof of Shearer's theorem - discussed in class.

(3) Say that a function $f : \{0,1\}^n \to \mathbb{R}$ is *increasing* if $f(x) \geqslant f(y)$ whenever $x_i \geqslant y_i$, for all coordinates i. We prove the following generalization of Harris' inequality as seen in class: Let $f, g : \{0,1\}^n \to \mathbb{R}$ be increasing functions and let $X_1, \ldots, X_n \sim \mathrm{Ber}\,(p)$ be independent. Then

$$\mathbb{E}\, f(X_1,\ldots,X_n)g(X_1,\ldots,X_n) \geqslant \mathbb{E}\, f(X_1,\ldots,X_n)\mathbb{E}\, g(X_1,\ldots,X_n). \tag{1}$$

Show that equation (1) implies Harris' inequality, as stated in class (about events in the probability space $G(n,p)$). Prove equation (1) by induction on N.
[Hint: Consider one variable functions of the form $\tilde{f}(y) = \mathbb{E}_{X_2,\ldots,X_n} f(y, X_2, \ldots, X_n)$.]

(4) The purpose of this exercise is to prove Janson's inequality, which says the following.
Let $\mathcal{H}$ be a hypergraph on vertex set V and, for $p \in [0,1]$, let $X \subset V$ be a p-random set. Define

$$\mu(\mathcal{H}) = \sum_{e \in \mathcal{H}} \mathbb{P}(X \supset e) \quad \text{and} \quad \Delta(\mathcal{H}) = \sum_{e,f \in \mathcal{H} : e \sim f} \mathbb{P}(X \supset e \cup f),$$

where we write $e \sim f$ if $e \neq f$ and $e \cap f \neq \emptyset$. Then we have

$$\mathbb{P}(X \not\supset \sigma, \text{ for all } \sigma \in \mathcal{H}) \leqslant e^{-\mu(\mathcal{H}) + \Delta(\mathcal{H})}.$$

(a) Write $\mathcal{H} = \{e_1, \ldots, e_m\}$ and define the events $A_i = \{e_i \subset X\}$. For $i \in [m]$ define

$$\mathcal{D}_i = \bigcap_{j : e_j \sim e_i} A_j^c \qquad \text{and} \qquad \mathcal{I}_i = \bigcap_{j < i : e_j \not\sim e_i} A_j^c.$$

For each $i \in [m]$, show that

$$\mathbb{P}\big(\mathcal{D}_i \big|\, A_i \cap \mathcal{I}_i\big) \geqslant \mathbb{P}\big(\mathcal{D}_i \big|\, A_i\big) \geqslant 1 - \sum_{j < i : i \sim j} \mathbb{P}\big(A_j \big| A_i\big)$$

(b) Finally use the above to prove

$$\mathbb{P}\big(A_i \,\big|A_1^c \wedge \cdots \wedge A_{i-1}^c\big) \geqslant \mathbb{P}(A_i) - \sum_{j < i : e_i \sim e_j} \mathbb{P}(A_i \wedge A_j).$$

(c) Now use the above to prove Janson's inequality.

(5) Let $G \sim G(n,p)$ where $p = \lambda/n$, and $\lambda > 0$ is fixed. Show that

$$\lim_{n \to \infty} \mathbb{P}\big(G \supset K_3\big) = e^{-\lambda^3/3}.$$

Now let $1/n \ll p \leqslant Cn^{-1/2}$. Show that

$$\mathbb{P}(G \not\supset K_3) = e^{-\Theta(p^3 n^3)}.$$

(6) There is also a very useful second form of Janson's inequality. Using the same assumptions as in the first question and $\Delta \geqslant \mu$, we have that

$$\mathbb{P}\big(X \not\supset \sigma, \text{ for all } \sigma \in \mathcal{H}\big) \leqslant \exp\left(-\frac{\mu(\mathcal{H})^2}{2\Delta(\mathcal{H})}\right).$$

Use the first Janson inequality to prove the second Janson inequality.
[Hint: apply the first part to a random subset of the events A_i, as defined above.]

(7) Show that if $n^{-1/2} \ll p \leqslant 1/2$ then

$$\mathbb{P}(G \not\supset K_3) = e^{-\Theta(pn^2)}.$$

(8) A set system $\mathcal{A} \subseteq \mathcal{P}([n])$ is said to be *intersecting* if for all $x, y \in \mathcal{A}$ we have $x \cap y \neq \emptyset$ Show that if $\mathcal{A}$ is intersecting then $|\mathcal{A}| \leqslant 2^{n-1}$. Show that if $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{P}([n])$ are such that $\mathcal{A}_i$ is intersecting for each i, then $|\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k| \leqslant 2^n - 2^{n-k}$.

(9) For $k \in \mathbb{N}$, $r \geqslant 3$, show that there exists $C > 0$ so that the following holds. If $p > Cn^{-2/r}$ and $G \sim G(n,p)$, then for every colouring of the vertices $c : V(G) \to [k]$, there exists a K_r where all its vertices have the same colour.