PCMI: RAMSEY THEORY ON GRAPHS - DAY 2

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(1) Prove the following lemma as seen in lecture. For $1/n \ll p \ll 1/2$, let $G \sim G(n,p)$. If $k \geqslant C(\log n)/p$, then with high probability, every set of k vertices in G induces at least $pk^2/16$ edges. Here $C > 0$ is an absolute constant.

(2) Prove that every graph on n vertices with maximum degree d contains an independent set of size $\geqslant n/(d+1)$. Is this sharp? Now show that for every graph

$$\alpha(G) \geqslant \sum_{x \in V(G)} \frac{1}{d(x)+1}.$$

(3) Let H be a 3-uniform hypergraph on n-vertices. The *degree* of a vertex v is defined by

$$d(v) = \big|\{e \in E(G) : v \in e\}\big|.$$

Say that $I \subset V(G)$ is *independent* if I does not contain any $e \in E(H)$. Let $\alpha(H)$ be the maximum size of an independent set. Show that if H has maximum degree d then

$$\alpha(H) \geqslant cn/d^{1/2},$$

for some constant $c > 0$. Is this sharp?

(4) Show that, for each $d \geqslant 3$ and $n \geqslant 2^d$, there exists a triangle-free graph G on n vertices with average degree $\geqslant d$ and

$$\alpha(G) \leqslant (2 + o(1))\frac{n}{d}\log d,$$

where the $o(1) = o_{d\to\infty}(1)$ term tends to 0 as $d \to \infty$.

(5) Let G be graph on n vertices with average degree $d \gg 1$ and with at most d^2n/λ^3 triangles, where $1 \ll \lambda \leqslant d$.

(a) Show that

$$\alpha(G) \geqslant \frac{cn}{d}\log \lambda,$$

for some $c > 0$.

(b) Now use this to show that

$$R(4,k) \leqslant C\frac{k^3}{(\log k)^2},$$

for some $C > 0$.

(c) Let H be a graph with $\mathrm{ex}(H,n) = O(n^{2-1/t})$. Show that

$$r(H,\dots,H,K_k) \leqslant C\left(\frac{k}{\log k}\right)^t,$$

where there are $r-1$ copies of H in the above, $C > 0$ is a constant that depends on H and the number of colours r.

(6) Let G be a n vertex graph with maximum degree d and with the property that $\chi(G[N(x)]) \leqslant k$ for all $x \in V(G)$. Show that there exists $c_k > 0$ so that

$$\alpha(G) \geqslant c_k \frac{n \log d}{d}.$$

(7) (+) Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that the following holds for all sufficiently large k. If $n > 2^{\varepsilon k}$ and $\chi : E(K_n) \to \{\text{red}, \text{blue}\}$ is a colouring where $\geqslant (1-\delta)\binom{n}{2}$ of the edges are blue. Then χ contains a monochromatic K_k.