Triangle-free graphs

1. (a) Derive an asymptotic formula for $|\mathcal{B}(n)|$, the number of bipartite graphs on n vertices. Determine the distribution of the sizes of the parts of the bipartition of a uniformly random bipartite graph.

   (b) For $p \gg \frac{\log n}{n}$, derive an asymptotic formula for the probability that $G(n,p)$ is bipartite, $\mathbb{P}_p[\mathcal{B}(n)]$.

2. Come up with a heuristic prediction for the following edge-density thresholds for a uniformly chosen K_{r+1}-free graph on n vertices and m edges

   (a) The threshold for being r-partite. (I.e. for $r = 2$, triangle-freeness, the answer is $\frac{\sqrt{3}}{4}\sqrt{\frac{\log n}{n}}$).

   (b) The threshold for the emergence of giant components of defect edges.

3. Let G be a 3-uniform, Δ-regular hypergraph on n vertices. Let $Z_G(\lambda)$ be its independence polynomial. Write out the first few terms of the cluster expansion for $\log Z_G(\lambda)$ in terms of subhypergraph counts of G.

4. (This example is from [1]). Let G_n be the following 3-uniform hypergraph on $n + \binom{n}{2}$ vertices. Let V_0 be a set of vertices labeled $1, \ldots, n$. For each pair $i, j \in V_0$ add one more vertex $v_{i,j}$ and form a 3-edge consisting of $i, j, v_{i,j}$.

   (a) What is the max degree of G_n.

   (b) Write an exact expression (with a sum) for the independence polynomial $Z_{G_n}(\lambda)$.

   (c) Prove an asymptotic upper bound on the largest disk around the origin in the complex plane free from zeros of Z_{G_n}.

5. (Percolation) Consider vertex percolation on an infinite hypergraph: every vertex is 'open' with probability p independently. The process percolates if there is an infinite connected component in the subhypergraph induced by the set of open vertices. Let $p_c(G)$ be the infimum over p so that the process percolates with probability 1.

   (a) Suppose G is k-uniform with maximum degree Δ. Prove an upper bound on $p_c(G)$.

   (b) Suppose G is k-uniform, linear (two edges overlap in at most one vertex), and has maximum degree Δ. Prove an upper bound on $p_c(G)$.

References

[1] S. Zhang. Hypergraph independence polynomials with a zero close to the origin. *Combinatorics, Probability and Computing*, pages 1–5, 2023.