Statistical physics and combinatorics

1. Let $i_k(G)$ be the number of independent sets of size k in a graph G.

   (a) Give a probabilistic interpretation (as, say, an expectation) for the quantity $\frac{i_{k+1}(G)}{i_k(G)}$ in terms of the uniform distribution over independent sets of size k in G.

   (b) Prove that for all G of maximum degree Δ on n vertices,

   $$\frac{i_{k+1}(G)}{i_k(G)} \geq \frac{n-(\Delta+1)k}{k+1}\,,$$

   and find a family of graphs for which the inequality is tight.

   (c) Use the above to prove that for all G of maximum degree Δ on n vertices,

   $$\frac{1}{n}\log Z_G(\lambda) \geq \frac{1}{\Delta+1}\log(1+(\Delta+1)\lambda)\,,$$

   and show that the inequality is tight. (Hint: recall that partition functions are multiplicative over disjoint graphs and that $Z_G(\lambda)$ is a polynomial).

2. Prove that for any $\lambda > 0$,

   $$i_k(G) = \frac{Z_G(\lambda)}{\lambda^k}\Pr_{G,\lambda}(|I|=k)$$

   where $i_k(G)$ is the number of independent sets of size k in G and the probability (and the partition function) is with respect to the hard-core model on G at fugacity λ.

3. Pick an independent set I from the hard-core model on a d-regular, triangle-free graph G at fugacity $\lambda > 0$, and pick v uniformly at random. Let Y be the number of uncovered neighbors of v with respect to I (an integer-valued random variable bounded between 0 and d).

   (a) Show that if Y is supported on 0 and d, then G is the complete bipartite graph $K_{d,d}$ or a union of $K_{d,d}$'s.

   (b) Suppose G has no component isomorphic to $K_{d,d}$. Prove a positive lower bound on $\Pr(Y \notin \{0,d\})$ in terms of d and λ. (The lower bound should not depend on the size of the graph G).

   (c) Using the previous two results and the theorem we proved in class, prove that for every d and $\lambda > 0$ there exists $\varepsilon > 0$ so that for every d-regular, triangle-free graph G on n vertices without a $K_{d,d}$ component,

   $$\frac{1}{n}\log Z_G(\lambda) \leq \frac{1}{2d}\log Z_{K_{d,d}}(\lambda) - \varepsilon\,.$$

   (The result extends to graphs that may contain triangles).

4. Gdenko's Local Central Limit Theorem states the following: Let X be an integer valued random variable with mean μ and variance σ^2 whose support has gcd 1. Let $X_1, X_2, \ldots$ be iid copies of X and let $S_n = \sum_{j=1}^{n} X_j$. Then for every integer k,

$$\Pr(X = k) = \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{(k-\mu)^2}{2\sigma^2 n}} + o(n^{-1/2}) .$$

(Why is the gcd condition necessary?)

Let $H_{d,n}$ be the graph that is a union of $n/2d$ copies of $K_{d,d}$.

(a) Show that for every $k \in \{0, 1, \ldots, n/2\}$ there exists $\lambda \geq 0$ so that the expected size of an independent set drawn from the hard-core model on $H_{d,n}$ at fugacity λ is exactly k.

(b) Fix $\varepsilon > 0$ and suppose $\varepsilon n < k < (1 - \varepsilon)n/2$. Choose λ so that the expected size size of an independent set drawn from the hard-core model on $H_{d,n}$ at fugacity λ is exactly k. Show that

$$\Pr(|I| = k) = \Theta(n^{-1/2}) .$$

(c) Prove the following. For every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon, d)$ large enough so that the following holds: for all $n \geq n_0$ divisible by $2d$, every d-regular G on n vertices that does not contain a component isomorphic to $K_{d,d}$, and every $\varepsilon n \leq k \leq n/2$,

$$i_k(G) < i_k(H_{d,n}) .$$

(Hint: use the result from Question 2 as an input)

(d) (Harder) Prove the same result without the assumption that G contains no component isomorphic to $K_{d,d}$. Hint: consider two cases, depending on the fraction of vertices in $K_{d,d}$ components. In the case that there are many $K_{d,d}$ components, analyze what the distribution of a uniformly random independent set of size k looks like restricted to the part of G that is not in a $K_{d,d}$ component.