PCMI 2023 XML: MEETING 2

ALBERT ARTILES

Problems

1. The goal of this problem is to understand tensor products. Let $\mathbb{F}$ be a field (Think of $\mathbb{F}$ as representing $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).

(1) Let V and W be vector spaces of $\mathbb{F}$. (What does this mean?), describe the *Cartesian product* of V and W, denoted by $V \times W$.

(2) Look up the definition of a bilinear map. Show that the dot product on $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is bilinear. Show that the determinant map from $\mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ given by $([s,t],[u,v]) \mapsto sv - tu$ is bilinear. What other bilinear maps can you come up with?

(3) Let L be the vector space whose basis set $V \times W$. This space is very big. How big?

(4) We create partition of L above. These are the rules: for all $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $c \in \mathbb{F}$, we have the following relations:
- $(v_1 + v_2, w_1) \sim (v_1, w_1) + (v_2, w_1)$
- $(v_1, w_1 + w_2) \sim (v_1, w_1) + (v_1, w_2)$
- $(cv_1, w_1) \sim c(v_1, w_1)$
- $(v_1, cw_1) \sim c(v_1, w_1)$.

$L/\sim$ is called the tensor product of V and W and is denoted $V \otimes W$. Show that $V \otimes W$ is a vector space over $\mathbb{F}$. The equivalence class of the vector (v_1, w_1) is denoted $v_1 \otimes w_1$.

(5) If V has dimension m and W has dimension n, what is the dimension of $V \otimes W$?

(6) Let Y be a vector space over $\mathbb{F}$. Define the map $\phi : V \times W \to V \otimes W$ by $(v,w) \mapsto v \otimes w$. Show that if $f : V \times W \to Y$ is a bilinear map, then there exists a (unique) linear map $\tilde{f} : V \otimes W \to Y$ such that $f = \tilde{f} \circ \phi$.

(7) Let Z be a vector space over $\mathbb{F}$ and $\psi : V \times W \to Z$ be a bilinear map. Suppose further that for each bilinear map $g : V \times W \to Y$, there exists a unique linear map $\tilde{g} : Z \to Y$ such that $g = \tilde{g} \circ \psi$. Show $V \otimes W$ and Z are isomorphic as vector spaces. (This shows that tensor products are intimately related. You have just proved what is called the universal property of tensor products)

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2. You might have encountered the group of 2×2 matrices with determinant 1 and real entries as a group that acts on $\mathbb{R}^2$. This group is usually called the special linear group and is denoted by $SL(2, \mathbb{R})$.

Let's think about the the action of this group on a different space. Let $\mathbb{H}$ denote the collection of complex numbers with *positive* imaginary part, that is $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$

(1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in $SL(2, \mathbb{R})$. Show that the following operation is a group action: $Az = \frac{az+b}{cz+d}$ on $\mathbb{H}$. Let K be the kernel of the action. $SL(2, \mathbb{R})/K$ is called the projective special linear group and is denoted as $PSL(2, \mathbb{R})$. Compute K.

(2) A hyperbolic line in $\mathbb{H}$ is either a ray orthogonal to $\mathbb{R} \subset \mathbb{C}$ or a circle orthogonal to $\mathbb{R} \subset \mathbb{C}$. Draw a few! Show that if $A \in SL(2, \mathbb{R})$ and l is a hyperbolic line, then Al is also a hyperbolic line.

(3) $SL(2, \mathbb{Z})$ is the subset of $SL(2, \mathbb{R})$ consisting of all matrices with integer entries. Show $SL(2, \mathbb{Z})$ a subgroup of $SL(2, \mathbb{R})$ and that it is *not* normal.

(4) $PSL(2, \mathbb{Z})$ is generated by the cosets $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} K$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} K$. Let s be the part of the unit circle in $\mathbb{H}$, compute the orbit of s under $PSL(2, \mathbb{Z})$. (Draw some pictures)

3. What does the collection of 2×2 unitary matrices look like?

(1) What are necessary and sufficient conditions for a matrix to be unitary?
(2) Show that the determinant of unitary matrix is either 1 or -1. The group $SU(2)$ is the subgroup of $U(2)$ with determinant 1. Is this group normal in $U(2)$? What is its index?
(3) The unit sphere in $\mathbb{C}^2$ is the collection of all complex tuples (z, w) such that $z\overline{z} + w\overline{w} = 1$. What can you tell me about this shape?
(4) If $A \in SU(2)$, then A looks like BLANK, what happens when you interpret the first collumn of A as a vector in $\mathbb{C}^2$. Is this procedure invertible?
(5) What do $SU(2)$ and $U(2)$ look like?