PCMI 2023 XML: MEETING 2

ALBERT ARTILES

Problems

1. Let $\alpha \in \mathbb{R}$. Let $T_\alpha : [0,1) \to [0,1)$ by $x \mapsto x + \alpha$ modulo 1.

- Describe the orbit of the point 0 under $T_{1/17}$.
- Describe the orbit of the point 0 under the $T_{\sqrt{2}}$.
- Consider the functions $A : [0,1) \to \mathbb{R}$ given by $x \mapsto \sin(x)$. Compute
$$\int_0^1 \sin(x)dx$$
.
- Using a computer compute the following limit:
$$\lim_{\mathbb{N}\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} A \circ T^k_{\sqrt{2}}(0).$$
What did you notice?
- Let $I = [1/3, 1/2)$. Consider the function $B : [0,1) \to \mathbb{R}$, where B is the indicator function on I. What do you think
$$\lim_{\mathbb{N}\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} B \circ T^k_{\sqrt{2}}(0).$$
is equal to?
- What can you say about T_α when $\alpha \in \mathbb{Q}$? What can you say about T_α when $\alpha \notin \mathbb{Q}$?

Something to think about: Look at the sequence of powers of 2: 1, 2, 4, 8, 16, 32, 64, 128... and from it build a new sequence by taking only the leading digit: 1, 2, 4, 8, 1, 3, 6, 1,...

- Does the number 7 ever appear in the new sequence?
- Can you approximate how many times 7 appears in the first 10^{100} terms of the sequence?
- Make a histogram to see how many time in the first 200 terms of the sequence does each of the numbers $1, 2, ..., 9$ appear in the sequence.

Date: July 2023.

2. Let's play a game: This is a two player game. Start with 5 crosses on a plane. Player 1 and Player 2 will take turns. A turn is as follows:

(1) Find two pointy bits from the crosses or dashes and connect them with a curve that does not intersect any other drawn curves.
(2) Draw a dash perpendicular to the the curve drawn somewhere along it. (This provides two more pointy bits)
(3) Next player's turn.

The first player who cannot make a move loses. Device a strategy to ensure you always win no matter how many crosses you start with!
Something to think about: What happens if we play this game not on a flat piece of paper, but on a sphere or on the surface of a doughnut or the surface of a pretzel or the projective plane. Does any thing change?

3. Let $\mathcal{A} = \{0, 1\}$. $\mathcal{A}^{\mathbb{N}}$ denotes the collection of infinite sequences in elements of the set $\mathcal{A}$. We define a the shift map

- $\sigma_{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is given by $\sigma(x)_i = x_{i+1}$. (this map deletes the first entry of the infinite sequence.)

Consider the subset of $\mathcal{A}^{\mathbb{N}}$ consisting of all sequence where you do not allow two 1's to appear next to each other, and denote this set by X.

- Show that $\sigma_{\mathbb{N}}(X) = X$.
- How many finite words of length 14 can an element of X start with?
- What happens if you interpret an element of X as the binary expansion of a real number between 0 and 1? What kind of numbers do you get out?

Do the same analysis above with the alphabet $\mathcal{B} = \{0, 1, 2\}$ and let X be the collection of all elements in $B^{\mathbb{N}}$ where one is not allowed to put a 1.

Something to think about: Can you construct a surjective, non-decreasing function $f : [0, 1) \to [0, 1)$ such that if f is differentiable at x, then $f'(x) = 0$?