*The beauty of mathematics only shows itself to more patient followers.–*
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# Problem Set 7

**PCMI USS, Summer 2023**

1. (a) Write the density matrix $\rho_1$ corresponding to the ensemble $\mathcal{E}_1 = \{(\frac{3}{4}, |0\rangle), (\frac{1}{4}, |1\rangle)\}$. Express in math (yellow) notaion.

   (b) Do the same for the density matrix $\rho_2$ of the ensemble $\mathcal{E}_2 = \{(\frac{1}{2}, \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle), (\frac{1}{2}, \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle)\}$.

   (c) What happens if you measure the mixed states from (a) and (b) in the computational basis? Think this through both (a) in terms of the ensembles and (b) in terms of the density matrices.

2. Let $N$ be a positive integer and consider the vector space $\mathbf{C}^N$ with orthogonal basis $|0\rangle, \ldots, |N-1\rangle$. The *shift operator* on $\mathbf{C}^N$ is the operator acting on a basis by

$$S|j\rangle = |j+1\rangle$$

   (Indices are taken modulo $N$. So $S|N-1\rangle = |0\rangle$.)

   (a) What does the matrix for $S$ look like? Is $S$ unitary? Is $S$ Hermitian?

   (b) What is the characteristic polynomial of $S$? What are the eigenvalues of $S$?

   (c) What are the eigenvectors of $S$? (Write each eigenvector so its first coordinate is 1.)

   (d) The eigenvectors of $S$ should look familiar. Let $F$ be the Fourier transform. Show that $FSF^\dagger$ is a diagonal matrix $D$ and find $D$. [Recall in general that if you take all the eigenvalues of a matrix $A$ and write these eigenvectors as columns of a matrix $C$, then $C^{-1}AC$ is diagonal.]

   (e) Describe the operator $S^{-1}$. What does $S^{-1}$ do to the standard basis and what is its matrix form? Show that $F$ also diagonalizes $S^{-1}$; that is, show that $FS^{-1}F^\dagger$ is diagonal and find its entries. [Use your equation from (d).]

   (f) Consider $T = \frac{1}{2}(S + S^{-1})$. Describe $T$ and again find the diagonal matrix $FTF^\dagger$.

   (g) Is $T$ Hermitian? What does that imply about the eigenvalues of $T$? Did you notice this in part (f)? Write these eigenvalues in terms of trig functions.

   (h) (For thought.) Take the state $|0\rangle$ and start repeatedly applying $T$. That is, for $k = 1, 2, 3, \ldots$ , let $v_k = T^k|0\rangle$. Compute $v_k$ for $k = 1, 2, 3$. Why does Mark Waldon think this is an interesting thing to do? What do you think happens as $k$ gets very big? What other questions about this situation come to mind? Can you answer any of them?

3. In class, we had 3 single qubit states that were equally separated by 120° angles, and we asked how well these states could be distinguished. We crushed it by finding a POVM that beats the best orthogonal measurement, and we even proved that our POVM is optimal. Now consider what happens when you have $n$ single qubit states that are equally separated by angles of $2\pi/n$.

4. (a) Suppose you have a mixed state defined by the ensemble $\{(p_i, |\psi_i\rangle)\}$ with corresponding density matrix $\rho$. Show that $\rho$ is a positive operator with trace 1.

   (b) Prove the converse. That is, if $\rho$ is a positive operator of trace 1, then there is an ensemble of states whose density matrix is equal to $\rho$. [Hint: Does the spectral theorem apply to $\rho$?]

5. (a) Let $\rho$ be the density matrix corresponding to an ensemble of states. Show that $\rho$ is a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

   (b) Consider the density matrix from Problem 1. Does this density matrix represent a pure state?

   (c) If $\rho$ is a density matrix that is not a pure state, what can you say about $\mathrm{Tr}(\rho^2)$?

6. Suppose $|\psi_i\rangle$, $i = 1, 2, 3$ are three single qubit states in the plane. Under what conditions is there is a POVM that can distinguish these states with probability $2/3$? Can you prove that your answer is correct?