We will always have STEM with us. Some things will drop out of the public eye and go away, but there will always be science, engineering, and technology. And there will always, always be mathematics.
– Katherine Johnson

Problem Set 3

PCMI USS, Summer 2023

1. (a) Show that the 2-qubit state $|\phi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$ is the same state as our favorite entangled EPR pair $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
   (b) Find another proof of this fact by computing $\langle\psi|\phi\rangle$.
   (c) Can you find any other expressions for this state?

2. Consider these two 2-qubit states. (You may remember these from Day 1.)

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)$$
$$|\psi_2\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle).$$

   (a) Is $|\psi_1\rangle$ a product state or an entangled state? Prove that your answer is correct.
   (b) Is $|\psi_2\rangle$ a product state or an entangled state? Prove that your answer is correct.
   (c) On the first problem set, I asked you to measure the first qubit $|\psi_1\rangle$ and determine the collapsed states. And same for $|\psi_2\rangle$. How are the answers to these questions related to your answers to (a) and (b)?
   (d) Consider a general 2-qubit state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$

   Under what conditions on the α_{ij} is $|\psi\rangle$ a product state?
   (e) Can you generalize your answer to (d)?

3. Consider the 3-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

   which is shared among Alice, Bob, and Charlie.
   (a) Is $|\psi\rangle$ a product state or an entangled state? Explain.
   (b) If Charlie measures his qubit, what are the possibilities for the collapsed state and their probabilities? Are these states entangled or unentangled?
   (c) How does this situation intuitively resemble the Borromean rings?

(d) The n-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\dots0\rangle + |1\dots1\rangle)$ is sometimes referred to as a *cat state*. Why does this name apply? Can you build a quantum circuit to create this state?

4. Now consider the 3-qubit state (called the W-state)

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

If Charlie measures his qubit in the computational basis, what is the probability that in the collapsed state $|\psi'\rangle$, Alice's and Bob's qubits are entangled?

5. Write the matrix for the following operators in the computational basis.

(a) $I \otimes X$

(b) $X \otimes I$

(c) $X \otimes Z$

(d) $H \otimes H$

(e) $H \otimes H \otimes H$

(f) What pattern do you see emerging in (d) and (e)? Can you express this pattern nicely in either math notation or Dirac notation?

6. Prove that if A and B are unitary operators, then so is $A \otimes B$.

7. Prove that if A and B are positive operators, then so is $A \otimes B$.

8. For each $a = (a_1, a_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, let $\mathcal{O}_a$ be the oracle operator in Deutsch's problem. Write down these 4 matrices explicitly.

9. In Deutsch's algorithm, we succeeded by using the query $|+\rangle|-\rangle$. Suppose you use the query $|-\rangle|-\rangle$ instead. Write down the four possible states you obtain after calling the oracle. With what probability can you now succeed in determining $a_0 \oplus a_1$?

10. If V is an inner product space, the set of unitary operators on V is denoted $U(V)$.

(a) Show that function

$$\rho : \mathbb{Z}_2 \times \mathbb{Z}_2 \to U(\mathbf{C}^2 \otimes \mathbf{C}^2)$$

with $\rho(a) = \mathcal{O}_a$ a representation of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. That is, prove that $\rho(a + a') = \rho(a)\rho(a')$ for all $a, a' \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Prove that any two of the matrices $\mathcal{O}_a$ and $\mathcal{O}_{a'}$ commute.

(c) Recall the key fact about commuting matrices: They can be simultaneously diagonalized! Find a basis of $\mathbf{C}^2 \otimes \mathbf{C}^2$ in which all four matrices are diagonal.

11. In Deutsch's problem, there were two bits (a_0, a_1), and the goal was to learn the sum $a_0 \oplus a_1$ of the two bits. Suppose that you want to learn the actual pair (a_0, a_1). You are given a single quantum query. With what probability can you succeed?