Numerical exploration in sphere packing, Fourier analysis, and physics

Exercises 3

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1. Find a decreasing potential function p for which $\mathbb{Z}$ is <u>not</u> a ground state in $\mathbb{R}$. (I.e., another configuration of density 1 has lower energy.)

(2) Fix a rapidly decreasing potential function p (e.g., a Gaussian), and let $E(\alpha)$ be a minimum energy for density α. (i.e., the inf of the lower energies for such configurations.)

Prove that $E(\alpha)$ is convex as a function of $1/\alpha$.
(Maxwell's theorem)

(Hint: try putting chunks at densities α_1 and α_2 next to each other to achieve a density in between.)

(3) Using Problem 2, show that the face-centered cubic lattice does not minimize energy for $r \mapsto e^{-\pi r^2}$ at density 1.

(This will require some explicit computation. It's best done w/ a computer.)

(4) You might guess that the 1d analogue of radial Fourier interpolation would be the following:

An even Schwartz fn.

$$f:\mathbb{R}\to\mathbb{R}$$

is uniquely det'd by

$f(n)$, $f'(n)$, $\hat{f}(n)$, and $\hat{f}'(n)$

for integers $n\geq 1$.

Show that this assertion is false.