EXERCISES AND SUPPLEMENTAL TOPICS FOR LECTURE 1

Unless otherwise stated, a *quadratic space* refers to a finite-dimensional vector space V over a field F of characteristic $\neq 2$, together with a quadratic form $q : V \to F$ such that the associated symmetric bilinear form is nondegenerate.

(1) Prove *Sylvester's theorem*: given a quadratic form (V, q) over $\mathbb{R}$, with a decomposition $(V, q) \simeq \langle 1\rangle^{\oplus r} \oplus \langle -1\rangle^{\oplus s}$, the integers r, s are invariants of (V, q) (i.e., are independent of the decomposition). In fact, r is the dimension of any maximal subspace of V on which q is positive definite. The difference $r - s$ is called the *signature* of the quadratic form.

(2) The *Cartan–Dieudonné theorem.* Let (V, q) be a quadratic space of dimension n. Any element of the orthogonal group $O(V, q)$ is a product of $\leq n$ reflections. For this, show that $O(V, q)$ acts transitively on the subsets $\{v \in V : q(v) = a\}$ for any $a \in k^\times$.

(3) Let (V, q) be a quadratic space. Then the direct sum $(V, q) \oplus (V, -q)$ is hyperbolic (i.e., isomorphic to a direct sum of copies of the hyperbolic plane $\langle 1, -1\rangle$).

(4) Let $\alpha, \beta \in F^\times$ with $\alpha^2 + \beta^2 \neq 0$. Then the quadratic form $(V, q) = \big\langle 1, -(\alpha^2 + \beta^2)\big\rangle$ has the property that the direct sum $(V, q) \oplus (V, q)$ is hyperbolic. However, V itself need not be hyperbolic (that's true if and only if $\alpha^2 + \beta^2$ is a square in F).

(5) Show that the collection of nonzero elements represented by the quadratic form $\langle 1, d\rangle$ (for any $d \in k^\times$) forms a subgroup of $k^\times$.

(6) Let $F \subset E$ be a field extension of finite degree. Let $\ell : E \to F$ be a nonzero F-linear map. Given any inner product $B(\cdot, \cdot)$ on an E-vector space V, show that the symmetric bilinear form $\ell(B(\cdot, \cdot))$ on the underlying F-vector space of V is nondegenerate (and hence defines an inner product).

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