Let $\mathcal{E}$ be a local field
(of char $\neq 2$).

<u>Def</u> The <u>Hilbert symbol</u>

$$(\cdot,\cdot)_{\mathcal{E}} : \mathcal{E}^{\times}/\mathcal{E}^{\times 2} \times \mathcal{E}^{\times}/\mathcal{E}^{\times 2} \longrightarrow \{\pm 1\}$$

is defined s.t.

$$(a,b)_{\mathcal{E}} = \begin{cases} 1 & \text{if } z^2 - ax^2 - by^2 \text{ has a nontrivial soln} \\ -1 & \text{otherwise} \end{cases}$$

<u>Properties of the Hilbert symbol</u>

1) $(a,b)_{\mathcal{E}} = (b,a)_{\mathcal{E}}$

2) $(a, -a)_{\mathcal{E}} = 1$

3) $(a, 1-a)_{\mathcal{E}} = 1$ $\quad a \in \mathcal{E}^\times \setminus \{1\}$.

$(z, x, y) = (1, 1, 1)$

3') $(a, b)_{\mathcal{E}} = (a, -ab)_{\mathcal{E}}$

$z^2 - ax^2 - by^2$

$\mathcal{E}$ $az^2 - a^2x^2 - aby^2$

$\mathcal{E}$ $a^2x^2 - az^2 + aby^2$

4) $(\cdot , \cdot)_{\mathcal{E}}$ is bilinear (of $\mathbb{F}_2$-vector spaces)

& in fact $\mathcal{E}^\times / \mathcal{E}^{\times 2} \times \mathcal{E}^\times / \mathcal{E}^{\times 2} \to \mathbb{F}_2$

& is nondegenerate.

Given $a \in \mathcal{E}^\times$, then

$(a,b)_{\mathcal{E}} = 1 \iff z^2 - ax^2 - by^2$ has a soln

$\iff b$ is in image of

$Nm: \mathcal{E}(\sqrt{a})^\times \longrightarrow \mathcal{E}^\times$

b/c $\quad b = Nm\left(\dfrac{z - \sqrt{a}\,x}{y}\right).$

LCFT: image of $Nm(\mathcal{E}(\sqrt{a})^\times \to \mathcal{E}^\times)$ is index 2 in $\mathcal{E}^\times$.

In general, proving bilinearity directly is not so easy, but can do it directly for $\mathcal{E} = \mathbb{Q}_p$ (by explicit computation).

Ex) $p > 2$.

$x, y \in \mathbb{Q}_p^\times$

$$x = p^a u \qquad a, b \in \mathbb{Z}$$
$$y = p^b v \qquad u, v \in \mathbb{Z}_p^\times$$

then

$$(x, y)_{\mathbb{Q}_p} = (-1)^{ab\,\varepsilon(p)} \left(\frac{\bar{u}}{p}\right)^b \left(\frac{\bar{v}}{p}\right)^a$$

where

$$\varepsilon(p) = \frac{p-1}{2}$$

$\bar{u}, \bar{v} \in \mathbb{F}_p^\times$ are reductions of u, v

and $\left(\frac{\quad}{p}\right)$ is Legendre symbol

(i.e. nontrivial map $\mathbb{F}_p^\times \to \pm 1$)

Subexample:

$(u,v)_{\mathbb{Q}_p} = 1.$

(b/c $z^2 - ux^2 - vy^2$ is isotropic by Hensel's lemma, since it's isotropic mod p)

Subexample:

$(pu, v)_{\mathbb{Q}_p} = \left(\frac{\bar{v}}{p}\right).$

b/c $z^2 - pu x^2 - vy^2$ is isotropic $\iff$ $z^2 - vy^2$ is isotropic $\iff$ $v \in \mathbb{Z}_p^\times$ is a square $\iff$ $\left(\frac{\bar{v}}{p}\right) = 1.$

By computing directly, follows that it is a nondegenerate bilinear map.

Ex) $\Sigma = \mathbb{R}$.

Then $(a,b)_{\mathbb{R}} = \begin{cases} -1 & \text{if } a, b < 0 \\ 1 & \text{otherwise.} \end{cases}$

$$\underset{\mathbb{Z}/2}{\underset{\shortparallel}{\mathbb{R}^\times/\mathbb{R}^2}} \times \underset{\mathbb{Z}/2}{\underset{\shortparallel}{\mathbb{R}^\times/\mathbb{R}^2}} \longrightarrow \{\pm 1\}$$

Ex) For $p = 2$, need to use classification of squares in $\mathbb{Q}_2$. Recall $x \in \mathbb{Z}_2^\times$ is a 2-adic square $\iff x \equiv 1 \mod 8$.

Suppose $x = 2^a u$
$y = 2^b v$

$$(x,y)_{\mathbb{Q}_2} = (-1)^{\varepsilon(u)\varepsilon(v) + a\,\omega(v) + b\,\omega(u)}$$

where
$$\varepsilon(u) = \frac{u-1}{2} \mod 2. \quad \left(\text{use } \mathbb{Z}_2 \to \mathbb{F}_2\right)$$
$$\omega(u) = \frac{u^2-1}{8} \mod 2.$$

Example:

$$(u,v)_{\mathbb{Q}_2} = (-1)^{\left(\frac{u-1}{2}\right)\left(\frac{v-1}{2}\right)}$$

$$(2,u)_{\mathbb{Q}_2} = (-1)^{\left(\frac{u^2-1}{8}\right)}$$

The Hilbert symbol is closely related to the structure of $W(\mathcal{E})$ ($\mathcal{E}$ local field).

Observation:

$(a,b)_{\mathcal{E}} = 1 \iff \langle 1, -a, -b \rangle$ is isotropic

$\iff \langle 1, -a, -b, ab \rangle = \langle 1, -a \rangle \otimes \langle 1, -b \rangle$ is hyperbolic

Fact: $W(\mathcal{E}) \supseteq I$ (even-dim forms)

$I^3 = 0$

$W(\mathcal{E}) \supseteq I \supseteq I^2 \supseteq I^3 = 0$

$gr^0 = W(\mathcal{E})/I = \mathbb{Z}/2$

$gr^1 = I/I^2 = \mathcal{E}^\times/\mathcal{E}^{\times 2}$

$gr^2 = \mathbb{Z}/2$ (unique anisotropic 4-dim $\mathcal{E}$).

$gr^1 \times gr^1 \to gr^2 = \mathbb{Z}/2$

$\mathcal{E}^\times/\mathcal{E}^{\times 2} \times \mathcal{E}^\times/\mathcal{E}^{\times 2}$ Hilbert symbol.

Def Let F be a field and let A be an abelian gp. A *symbol* on F w/ values in A is a bilinear map (of $\mathbb{Z}$-modules)

$$\varphi : F^\times \times F^\times \longrightarrow A$$

s.t. $\varphi(a, 1-a) = 0 \quad a \notin \{0,1\}$

(Axiomatizes some of the properties of the Hilbert symbol)

Fact: $\varphi(a, -a) = 0$.

Use $-a = \dfrac{1-a}{1-a^{-1}}$

$$\varphi(a, -a) = \varphi\left(a, \frac{1-a}{1-a^{-1}}\right)$$

$= \varphi(a, 1-a) - \varphi(a, 1-a^{-1})$

$= \varphi(a, 1-a) + \varphi(a^{-1}, 1-a^{-1})$

$= 0 + 0 = 0.$

Fact $\varphi(a, b) = -\varphi(b, a).$

antisymmetry.

$0 = \varphi(ab, -ab)$ (by above)

$\cancel{\varphi(a, -a)} + \varphi(a, b) + \underbrace{\varphi(b, -a)} + \varphi(b, b)$

$= \varphi(a, b) + \varphi(b, a) + \varphi(b, -1) + \varphi(b, b)$

$= \varphi(a, b) + \varphi(b, a) + \cancel{\varphi(b, -b)}$

$\Rightarrow \varphi(a, b) = -\varphi(b, a)$

For the purposes of quadratic forms, main interest is in symbols w/ values in $\mathbb{F}_2$-vector spaces (such as the Hilbert symbol).

Given a symbol, there is a natural way to extract invariants of quadratic forms.

<u>Def</u> Let $\varphi: F^\times \times F^\times \to A$ be a symbol, suppose A is an $\mathbb{F}_2$-vector space (write multiplicatively)

Then can use the symbol φ to define an <u>invariant</u> of any quadratic form over F;

$\varepsilon_\varphi \in A$.

Here
$$\varepsilon_\varphi(\langle a_1, \dots, a_n \rangle) = \prod_{i<j} \varphi(a_i, a_j)$$

Need to show: this is independent of diagonalization.

Recall from last week that there are the following isos

$$\langle a, b \rangle \cong \langle b, a \rangle$$
$$\langle a, b \rangle \cong \langle a, bu^2 \rangle$$
$$\langle a, b \rangle \cong \left\langle a+b, \frac{ab}{a+b} \right\rangle.$$

(depend on isos of 2-d forms),

$a+b \neq 0$.

Need: if $\langle a, b \rangle \cong \langle c, d \rangle$ then $\varphi(a, b) = \varphi(c, d)$.

In fact,

$$c = ax^2 + by^2$$

for example (wlog) $\quad x = 1,\ y = 1$

$$c = a + b$$

$$\frac{a}{c} + \frac{b}{c} = 1$$

$$\Rightarrow \varphi\left(\frac{a}{c}, \frac{b}{c}\right) = 1$$

$$\varphi(a,b)\,\varphi(a,c)\,\varphi(b,c)\,\varphi(c,c) = 1$$

$$\Rightarrow \varphi(a,b)\,\varphi(\underset{"}{abc}, c) = 1$$

$abc = d$ up to squares.

$$\Rightarrow \varphi(a,b) = \varphi(c,d).$$

Example If E a local field of char $\neq 2$

$\Rightarrow$ define for each quadratic form V an invariant in $\{\pm 1\}$ by the above construction applied to Hilbert symbol.

$\varepsilon(V) \in \{\pm 1\}$. (called "Hasse invariant").

$$\varepsilon(V \oplus V') = \varepsilon(V) \cdot \varepsilon(V') \cdot (\det V, \det V')_E$$

Theorem:
If E is any local field of char $\neq 2$, then quadratic forms over E are isomorphic

$\iff$ 1) same dim

2) same det $\in E^\times / E^{\times 2}$

3) same Hasse invariant in $\{\pm 1\}$.

Main pt: - Quadratic forms of dim ≥ 5 are isotropic

- Unique anisotropic form of dim 4.

If (V, q) and (V', q') have the same invariants i want:

$(V \oplus V', q \oplus -q')$ is hyperbolic.

same invariants as a hyperbolic form of right dimension.